Coxeter & Schubert
Combinatorics of μ-Involutions

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joint with Zach Hamaker

5.8.2026

LACIM . UQAM

Outline

Coxeter

- μ-involutions
- Weak/Strong Bruhat order
- Atoms

Schubert

- Variety of Complete Quadrics
- Schubert polynomials
- Transition Equation

Background

Involution : $625431 \quad (w^{-1} = w)$

Cycle notation : $(6,1)(5,3)(2)(4)$

Composition : $(1,4,1,3) = \mu \vDash 9$

Background

Involution: $625431 \quad (w^{-1} = w)$

Cycle notation: $(6,1)(5,3)(2)(4)$

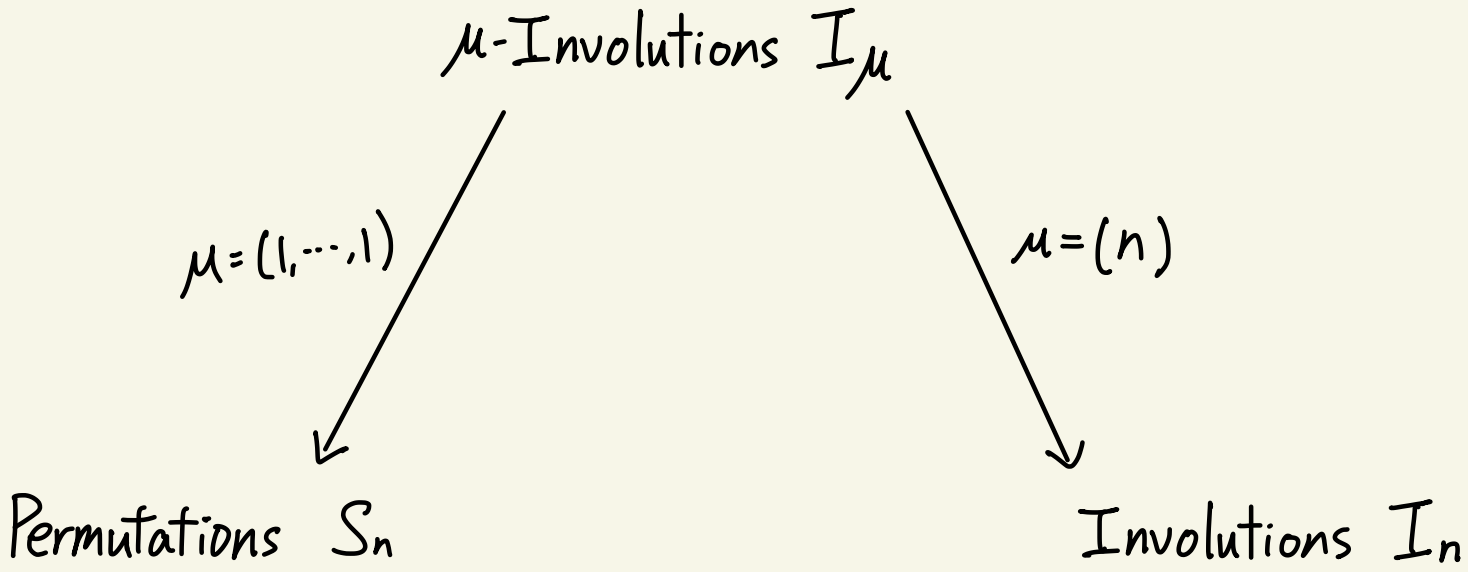
Composition: $(1,4,1,3) = \mu \vDash 9$

μ-Involution: $4\,|\,5317\,|\,8\,|\,962$

Standardization: $1\,|\,3214\,|\,1\,|\,321$

μ-Cycle notation: $(4)\,|\,(5,1)(3)(7)\,|\,(8)\,|\,(9,2)(6)$

Background


μ-Involutions I_μ
$\mu=(1,\cdots,1)$
$\mu=(n)$
Permutations S_n
Involutions I_n

Weak Order & Atoms

Weak Order Permutations S_n

generators s_i for $i \in [n-1]$

Relations:

$$s_i^2 = e$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i s_j = s_j s_i$$

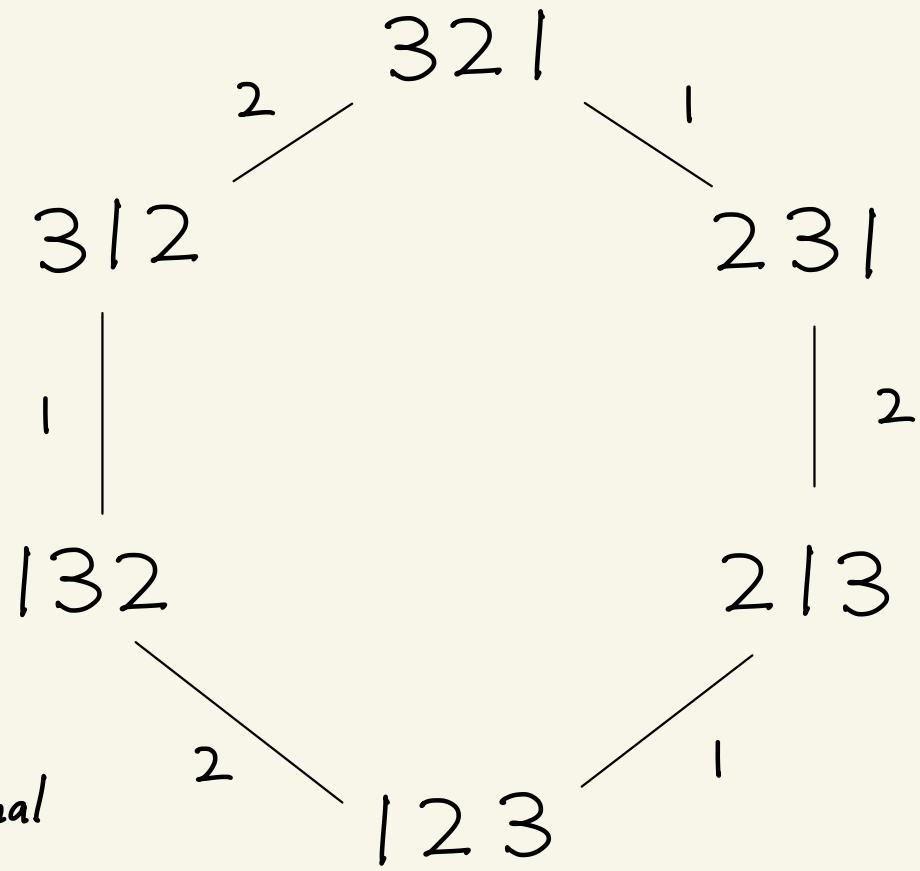
for $|i-j| > 1$

Words:

$231 = s_1 s_2 = s_1 s_2 s_2 s_2$

Reduced words : length minimal

$R(231) = \{s_1 s_2\}$



Weak Order μ-Involutions

Generators s_i for $i \in [n-1]$

swaps <u>value</u> i and $i+1$

Different blocks

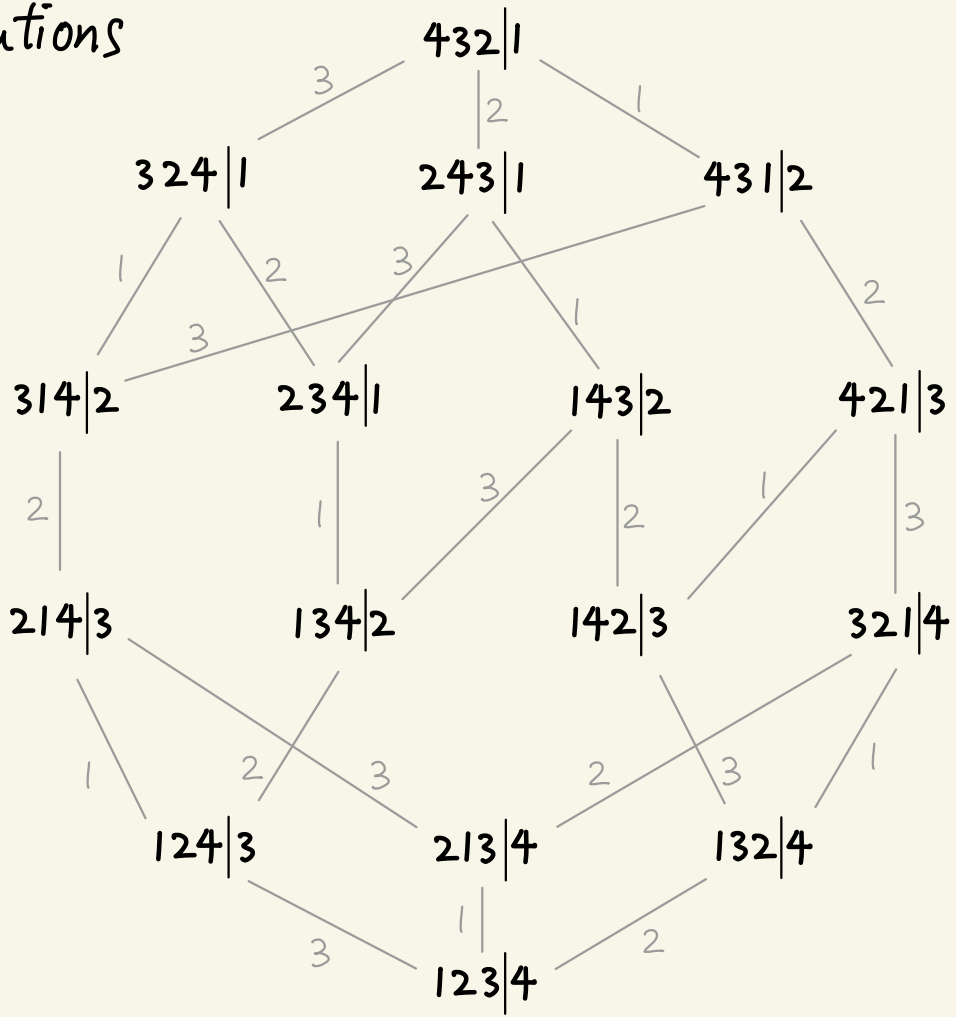
$3 \mid 4 \longrightarrow 4 \mid 3$

Both fixed points

$\mid 3\ 4 \mid \longrightarrow \mid 4\ 3 \mid$

Neither

$\mid 3 \mid 4 \mid \longrightarrow \mid 4\ 3 \mid 1 \mid$



Weak Order μ-Involutions

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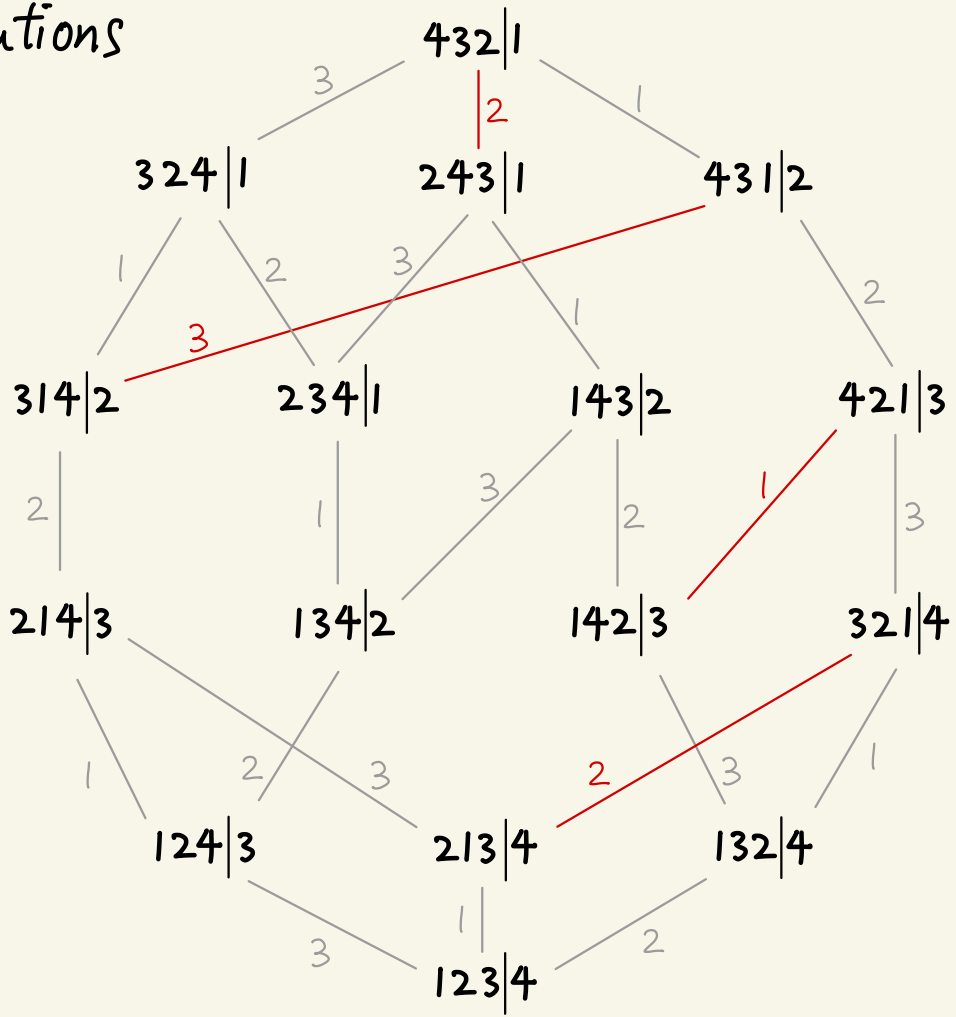
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Weak Order

μ-Involutions

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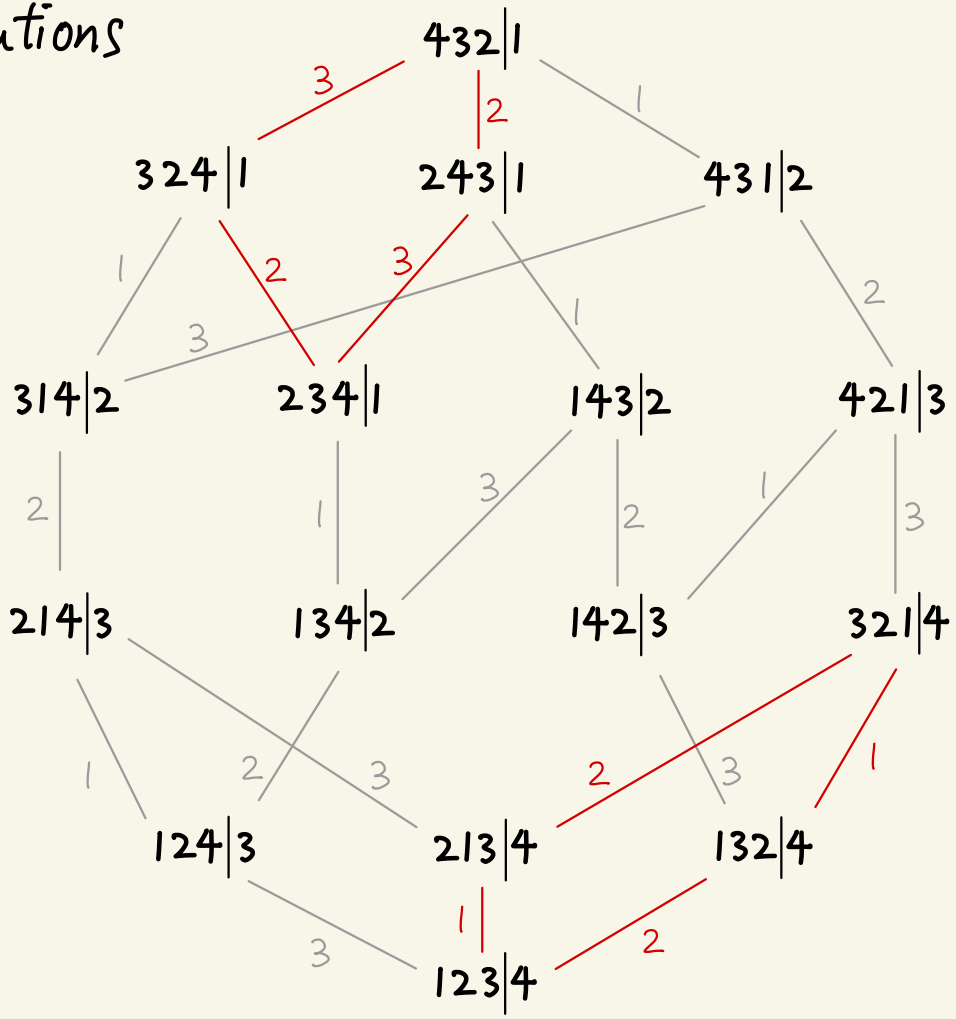
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$s_i s_j = s_j s_i$$

for $|i-j| > 1$

Sometimes

$$\pi \cdot s_i s_{i+1} = \pi \cdot s_{i+1} s_i$$



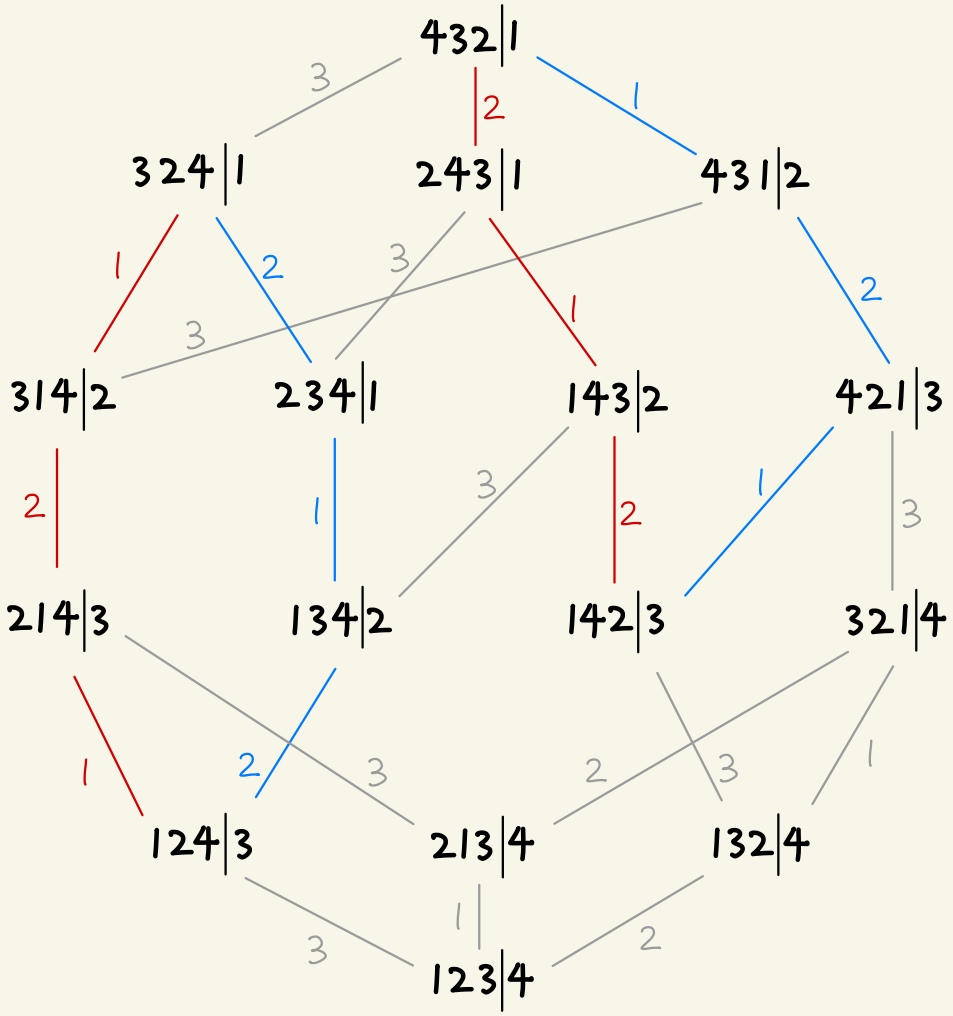
Weak Order

$\mu \vDash n$, I_μ has generators

s_i for $i \in [n-1]$

Relations

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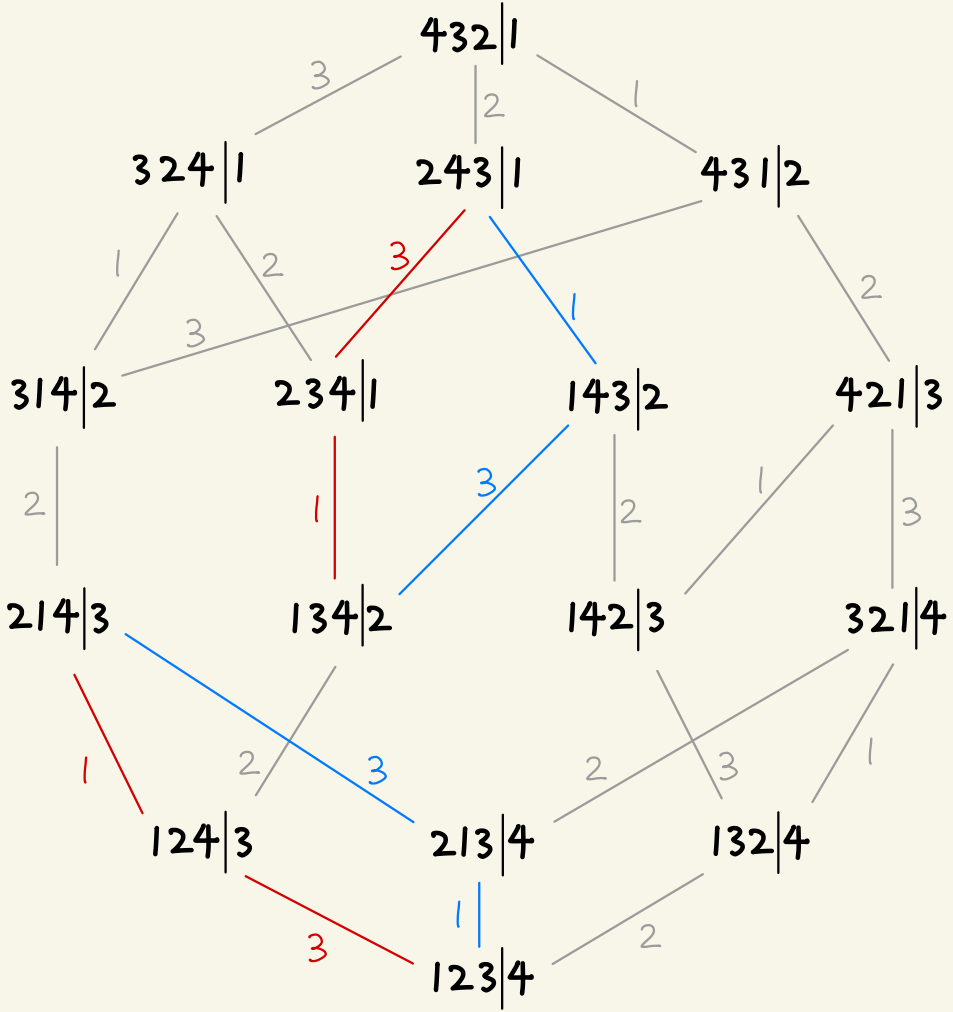
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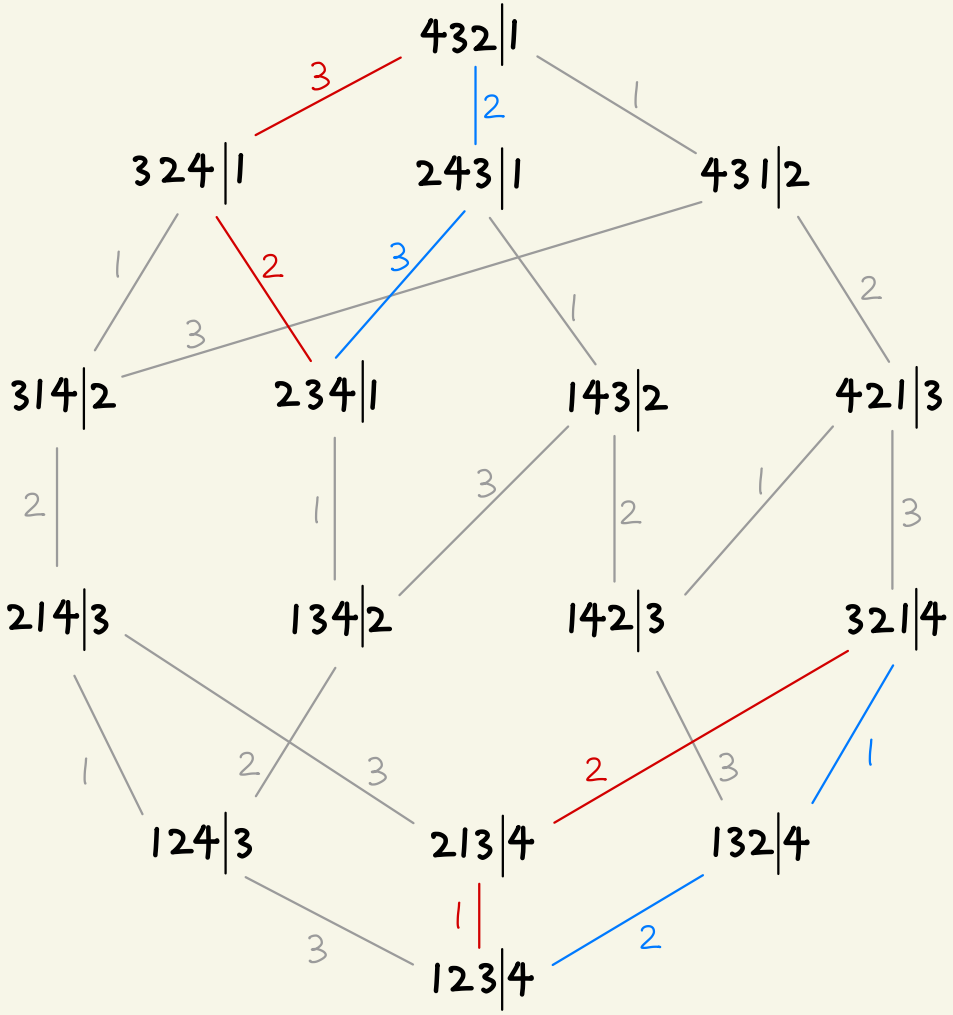
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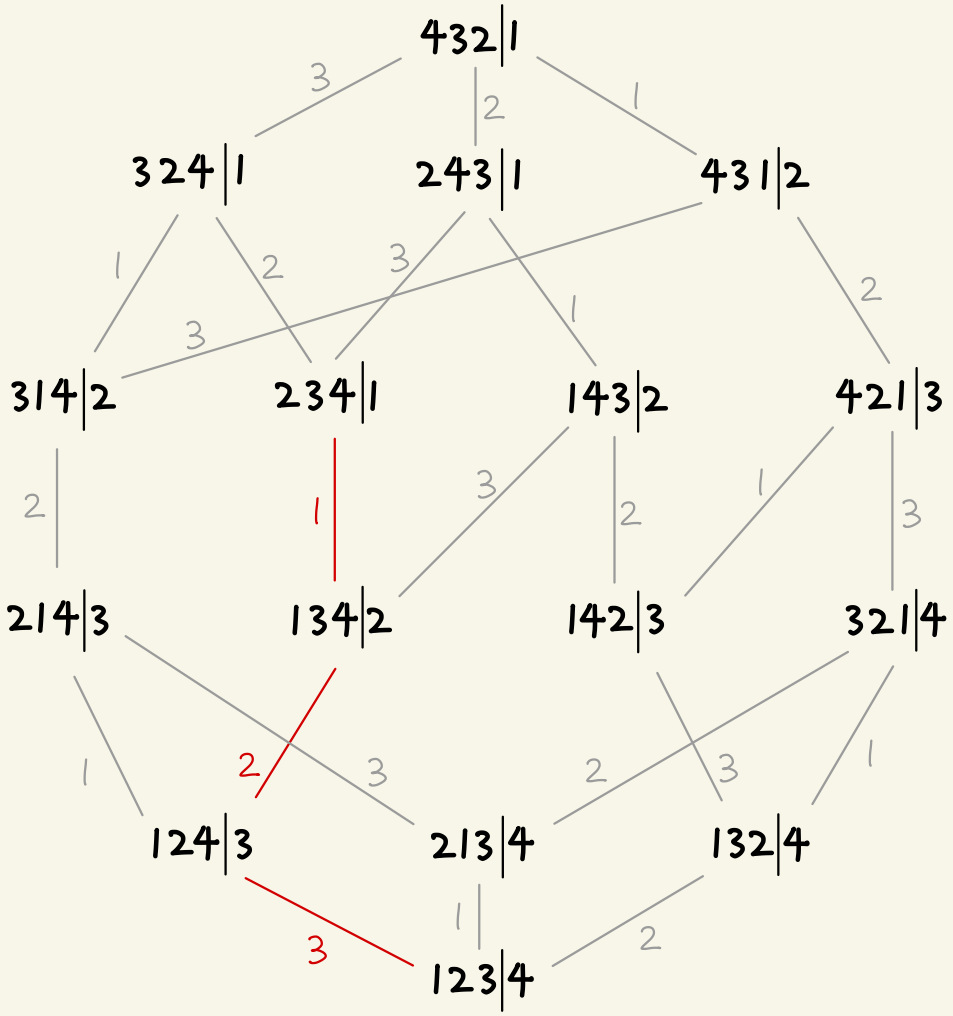
$$\pi \cdot s_i s_{i+1} = \pi \cdot s_{i+1} s_i$$



Words

$\pi \in I_\mu$, $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$

$234|1 = s_3 s_2 s_1 = \overline{321}$



Words

$\pi \in I_\mu$, $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$

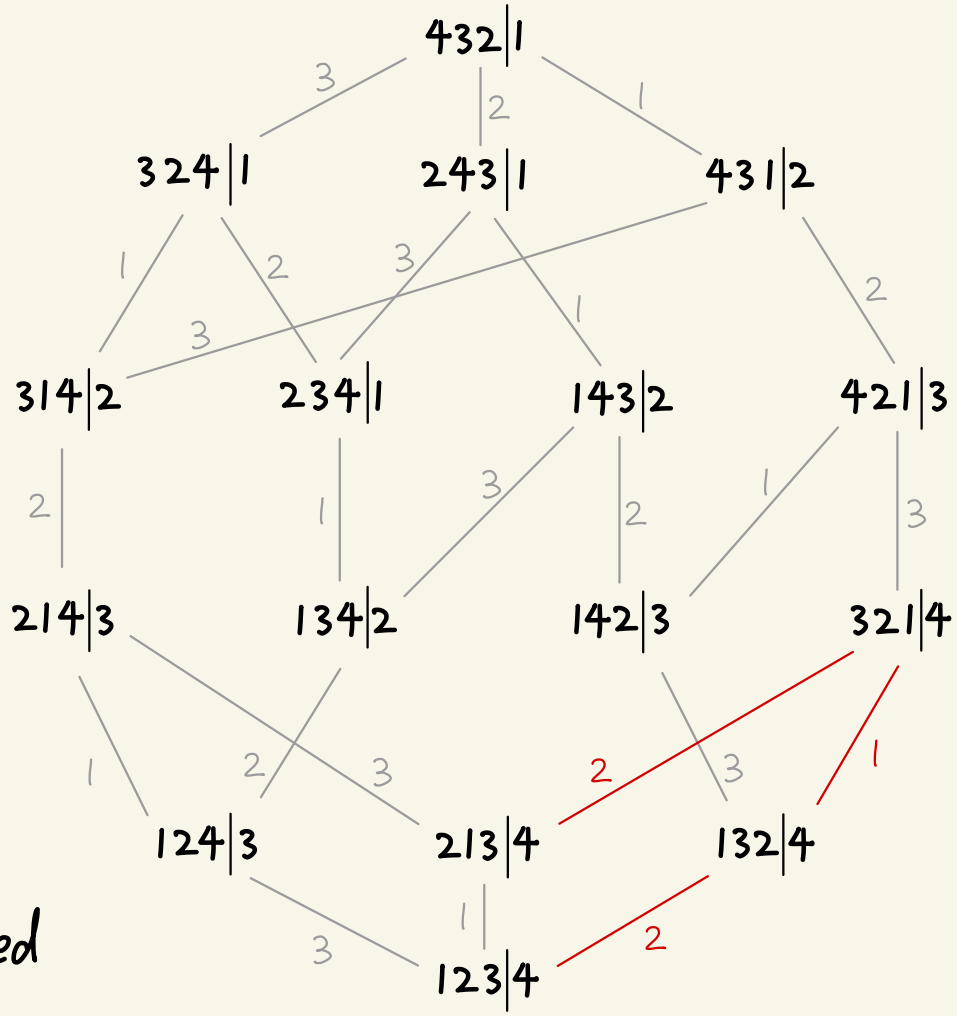
$234|1 = s_3 s_2 s_1 = \overline{321}$

A word is <u>μ-reduced</u> if k is minimal.

RMK:

μ-reduced $\Rightarrow$ reduced

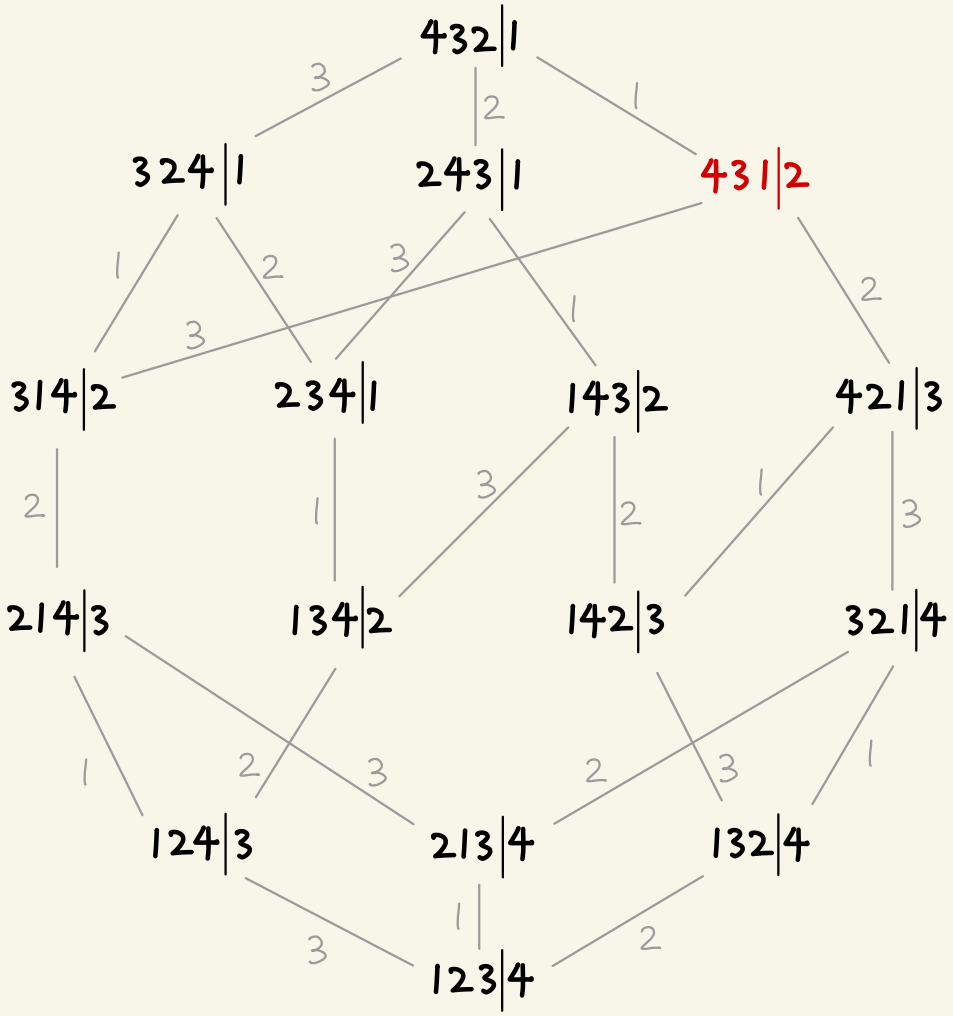
Ex. $\overline{212}$ is not μ-reduced



μ-Reduced Words

(Coxeter) Length is

$\ell(\pi) =$ length of its μ-reduced words



μ-Reduced Words

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Ex. $\pi = 431|2$

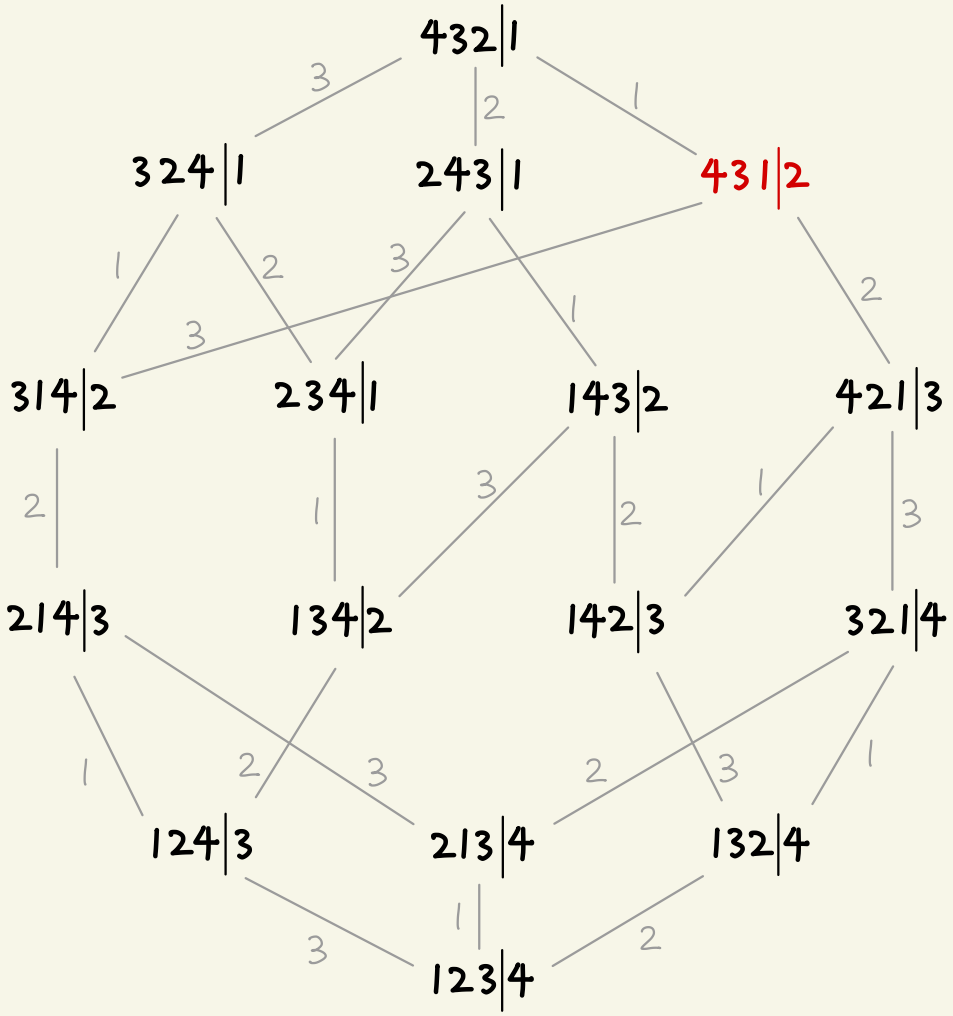
5 length of underlying permutation

2 inversions between blocks

4 n

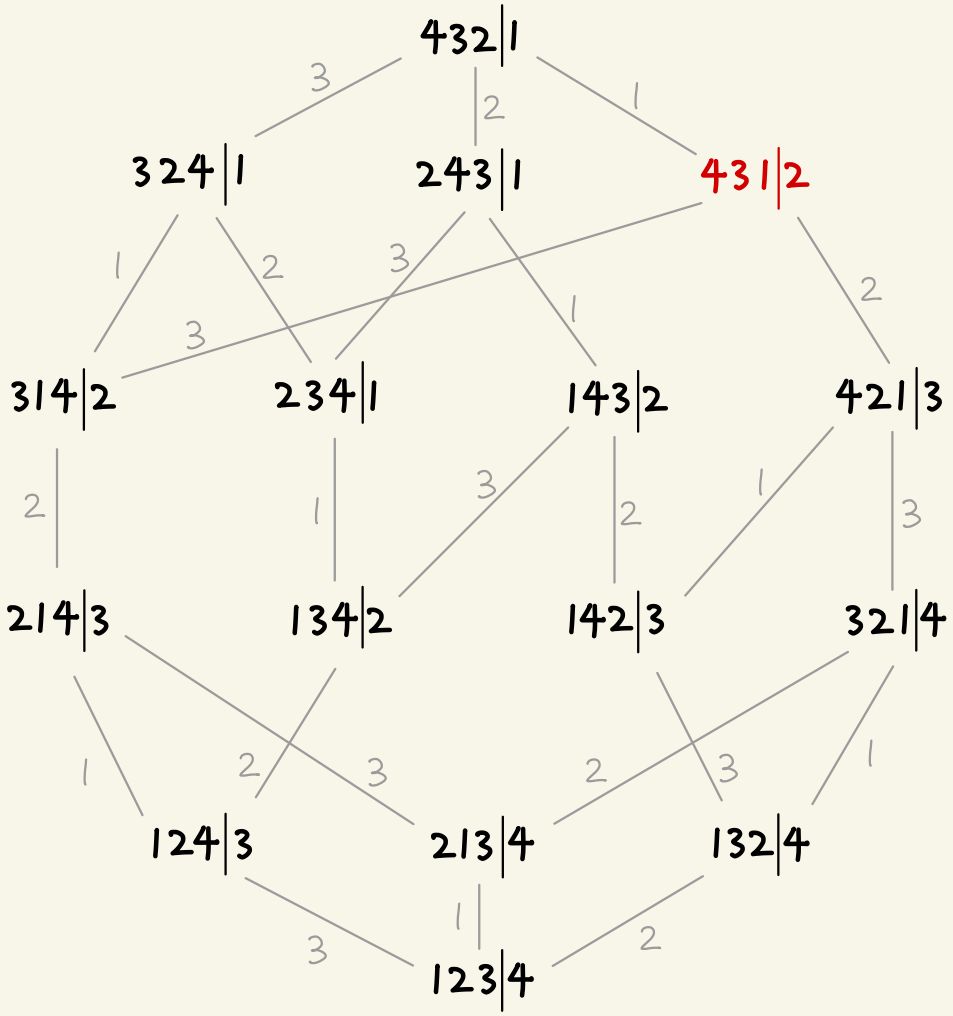
3 # of cycles

$\ell(\pi) = \frac{1}{2}(5+2+4-3) = 4$



Atoms

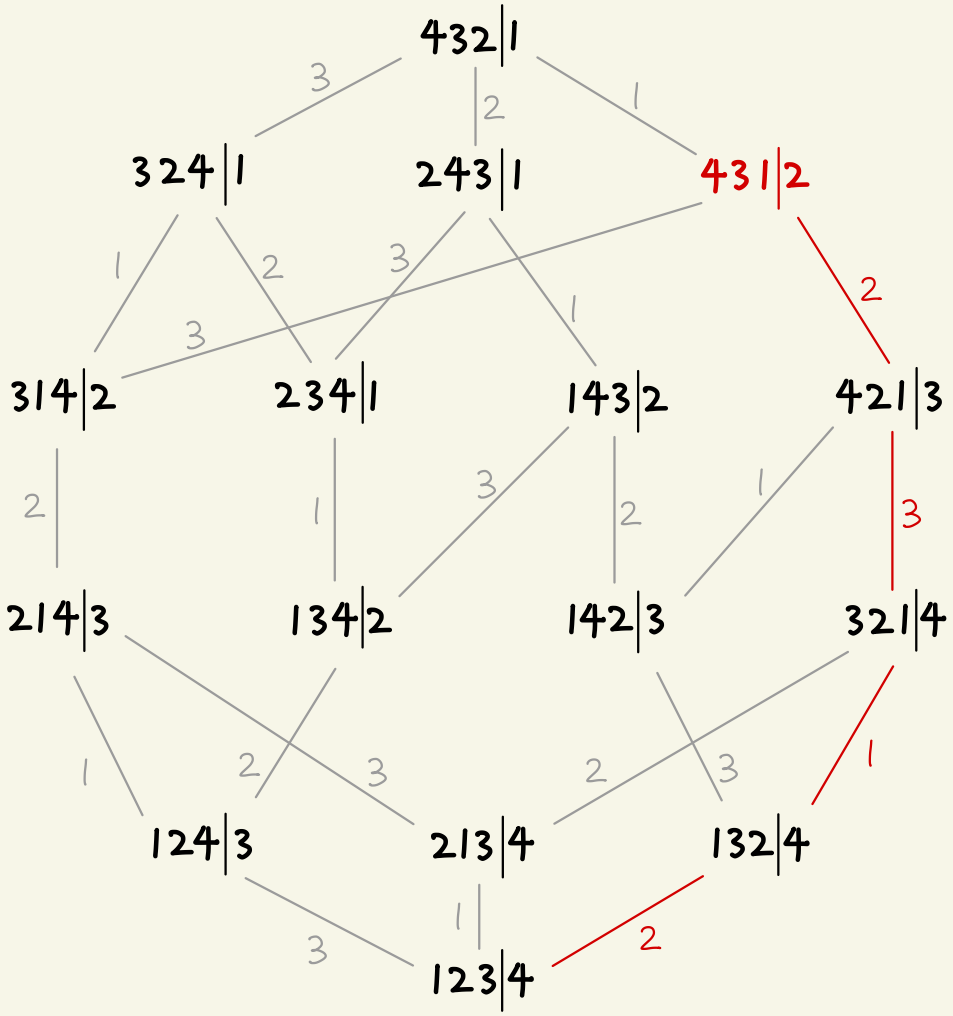
$R(\pi) = \left\{ \begin{array}{l} \mu\text{-reduced} \\ \text{words of } \pi \end{array} \right\}$



Atoms

$R(\pi) = \{ \mu\text{-reduced words of } \pi \}$

2132

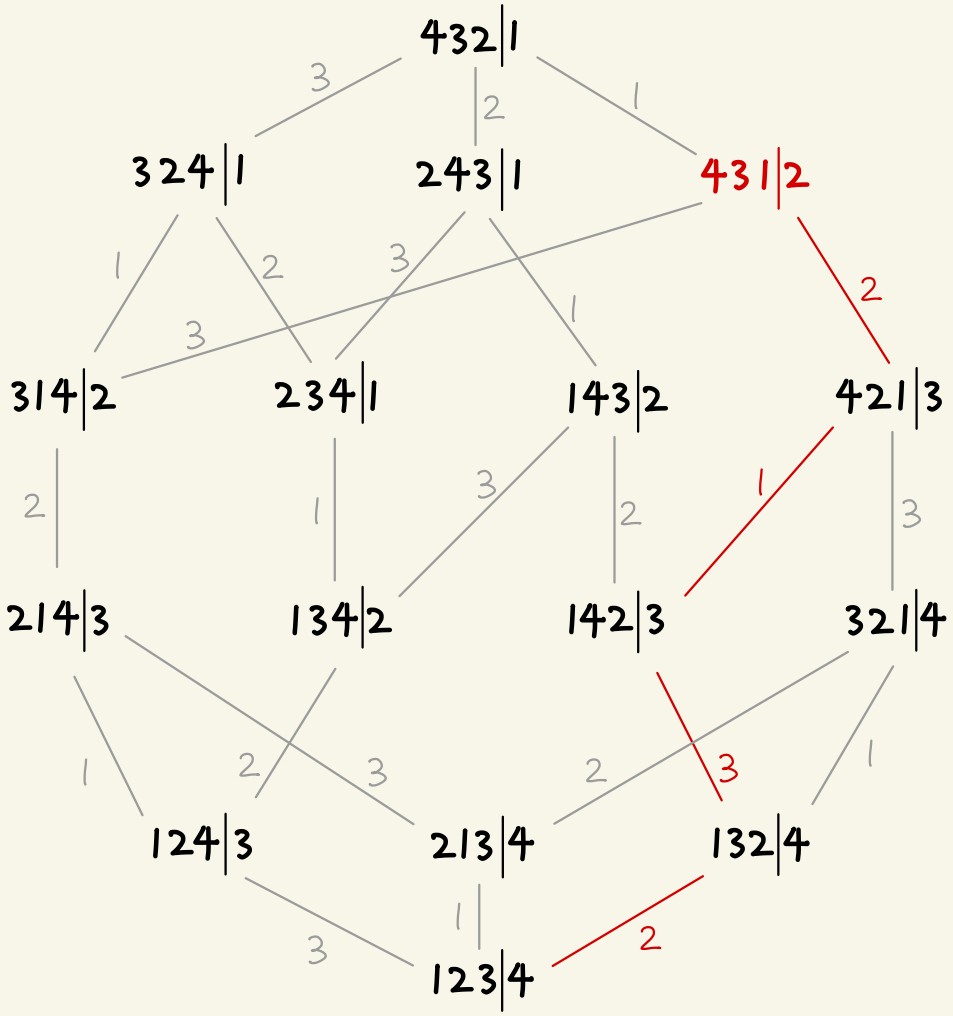


Atoms

$R(\pi) = \{ \mu\text{-reduced words of } \pi \}$

2 1 3 2

2 3 1 2



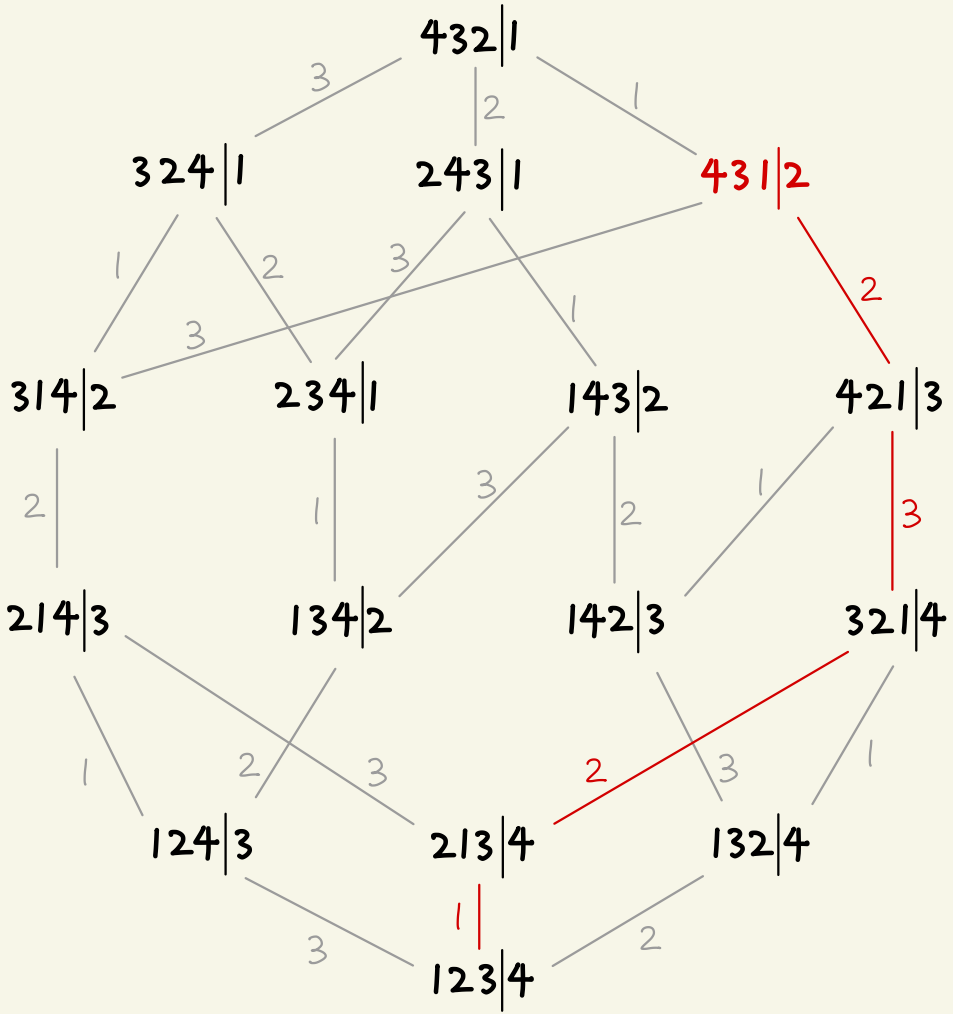
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Atoms

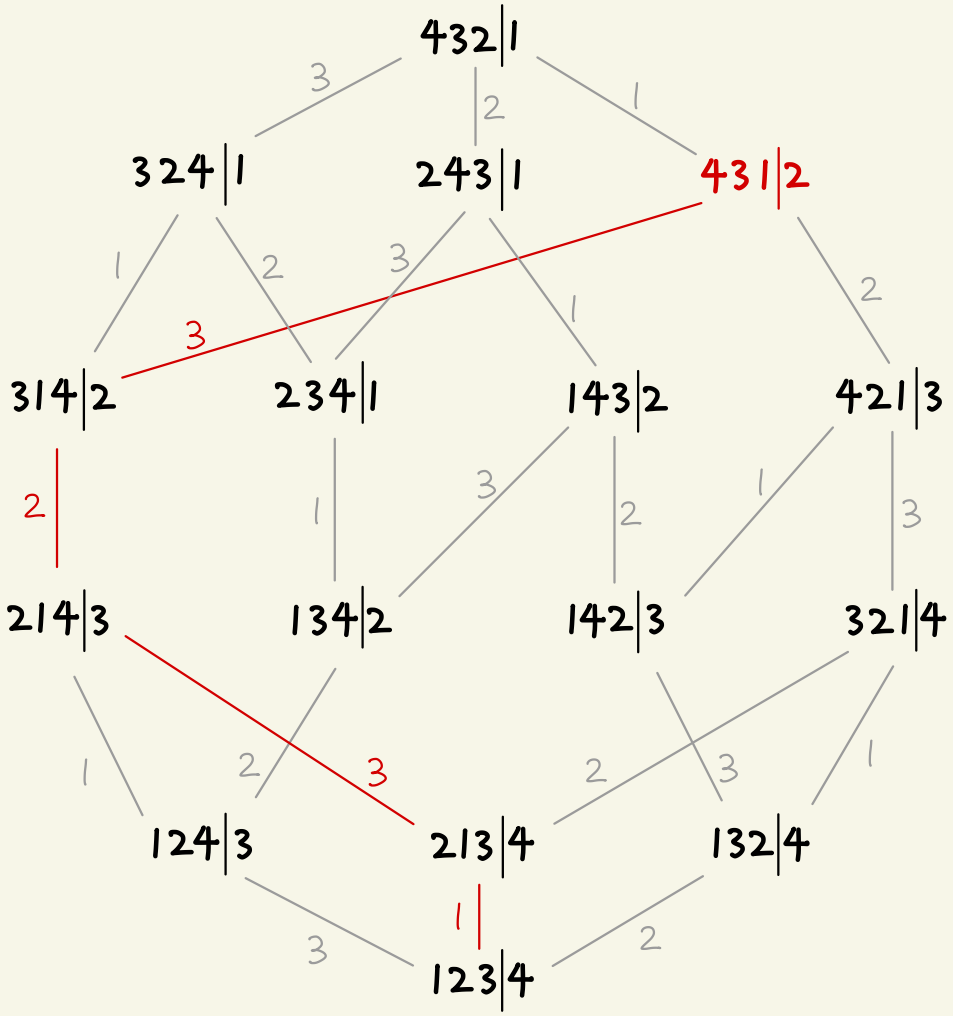
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2 1 3 2

2 3 1 2

1 2 3 2

1 3 2 3



Atoms

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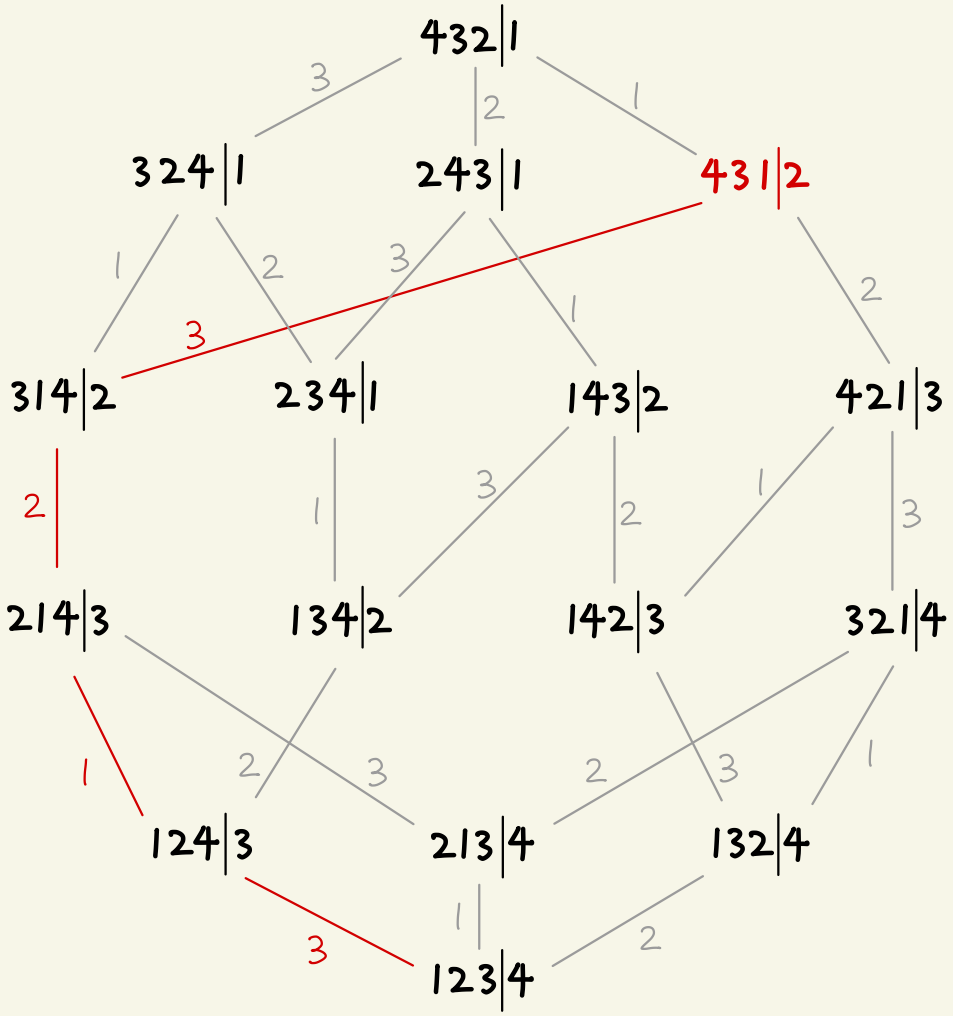
2 1 3 2

2 3 1 2

1 2 3 2

1 3 2 3

3 1 2 3



Atoms

$R(\pi) = \{ \mu\text{-reduced words of } \pi \}$

$\overline{2132}$

$\overline{2312}$

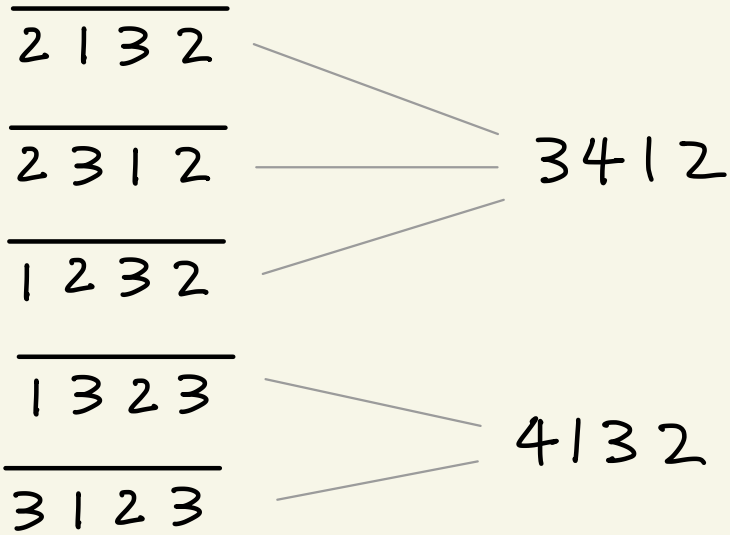
$\overline{1232}$

$\overline{1323}$

$\overline{3123}$

Atoms

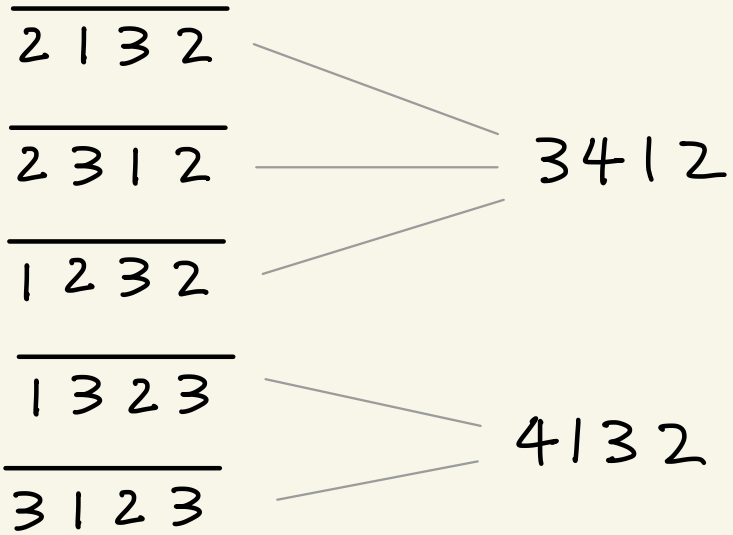
$R(\pi) = \{ \mu\text{-reduced words of } \pi \}$



Atoms

$$R(\pi) = \{\mu\text{-reduced words of } \pi\} = \bigcup_{w \in A_\mu(\pi)} R(w)$$

$A_\mu(\pi)$ called the set of <u>atoms</u> of π.



$A_\mu(\pi)$ gives G_w expansion of G_π^μ.

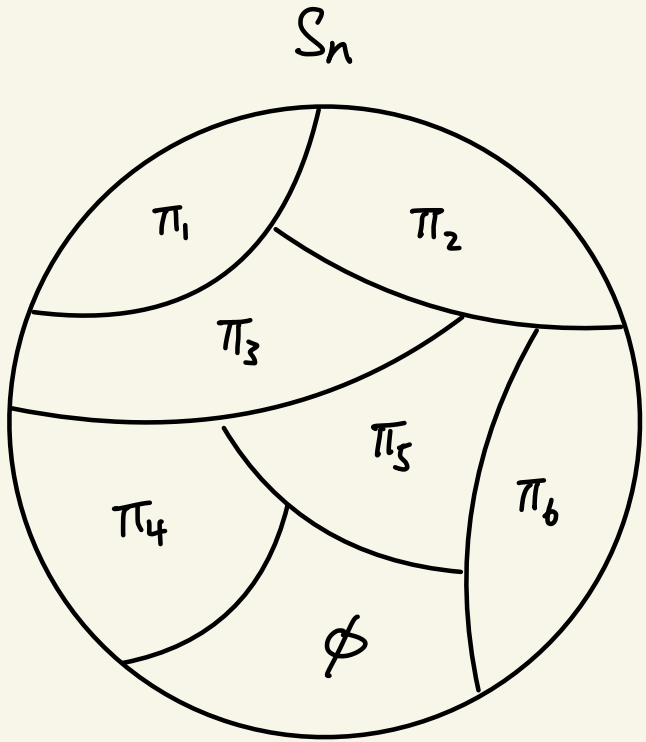
Goal: Give a combinatorial description of $A_u(\pi)$.

The set of inverse atoms, $A_u^{-1}(\pi) = \{w^{-1} : w \in A_u(\pi)\}$,

has nice a combinatorial description.

RMK: For a fix $\mu \vDash n$,

- $A_\mu(\pi_i)$ are pairwise disjoint.
- Some permutations are not atoms of any μ-Involution.



Involution Inverse Atoms

Thm [Can-Joyce-Wyser] [Hamaker-Marberg-Palowski]

Main idea: cycles dictate the relative orders of letters in any inverse atom.

Cycles	Order	Subword in inverse atom
(a,b)	$a > b$	ab
$(a,b)(c,d)$	$a > c$, $b > d$	$cdab$
	$a > c > d > b$	$cdab$ $cabd$ $abcd$
$(a,b)(c)$	$a > c > b$	cab abc

Ex. $\pi = 5327164$ $(5,1)(3,2)(7,4)(6)$

$(5,1)(3,2) \Rightarrow 3251, 3512, 5132$

$(5,1)(7,4) \Rightarrow 5174$ $\quad (5,1)(6) \Rightarrow 516$

$(3,2)(7,4) \Rightarrow 3274$ $\quad (3,2)(6) \Rightarrow 326$

$(7,4)(6) \Rightarrow 674, 746$

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$$A_{(n)}^{-1}(\pi) = \{ 3251674 \}$$

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$$A_{(n)}^{-1}(\pi) = \left\{ \begin{matrix} 3251674 \\ 3512746 \end{matrix} \right\}$$

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$$A_{(n)}^{-1}(\pi) = \left\{ \begin{matrix} 3251674 & 3251746 & 3512674 \\ 3512746 & 5132674 & 5132746 \end{matrix} \right\}$$

μ-Involution Inverse Atoms

Thm [C.-Hamaker] For $\pi \in I_\mu$ with blocks $B_1 \cdots B_k$,

$$A_\mu^{-1}(\pi) = \left\{ Q_1 Q_2 \cdots Q_k : Q_i \in A^{-1}(B_i) \right\}$$

μ-Involution Inverse Atoms

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Ex. $\pi = 651|3|742$

Block	$B_1 : 651$	$B_2 : 3$	$B_3 : 742$
Cycles	$(6,1)(5)$	(3)	$(7,2)(4)$
$A^{-1}(B_i)$	$561, 615$	3	$472, 724$

$$A_\mu^{-1}(\pi) = \left\{ 5613472,\ 5613724,\ 6153472,\ 6153724 \right\}$$

Something Interesting ...

$\pi = 651|3|742 \qquad (6,1)(5)|(3)|(7,2)(4)$

$61537\underline{24} \in A_\mu^{-1}(\pi) \qquad \underline{26}47315 \in A_\mu(\pi)$

Cycle notation gives an inverse atom.

Strong Bruhat Order

Strong Bruhat Order

$\pi \leq^{\mu} \tau$ orbit closure containment

Thm [Timashev]

weak order $\Rightarrow$ Strong Bruhat Order

Thm [Banerjee-Can-Joyce]

For $\pi, \tau \in I_\mu$, $\pi \leq^{\mu} \tau$ if and only if $\forall\, r \in R(\tau)$, $\exists\, r' \in R(\pi)$ s.t. r' is a subword of r and $\ell(\pi)+1 = \ell(\tau)$.

Ex.

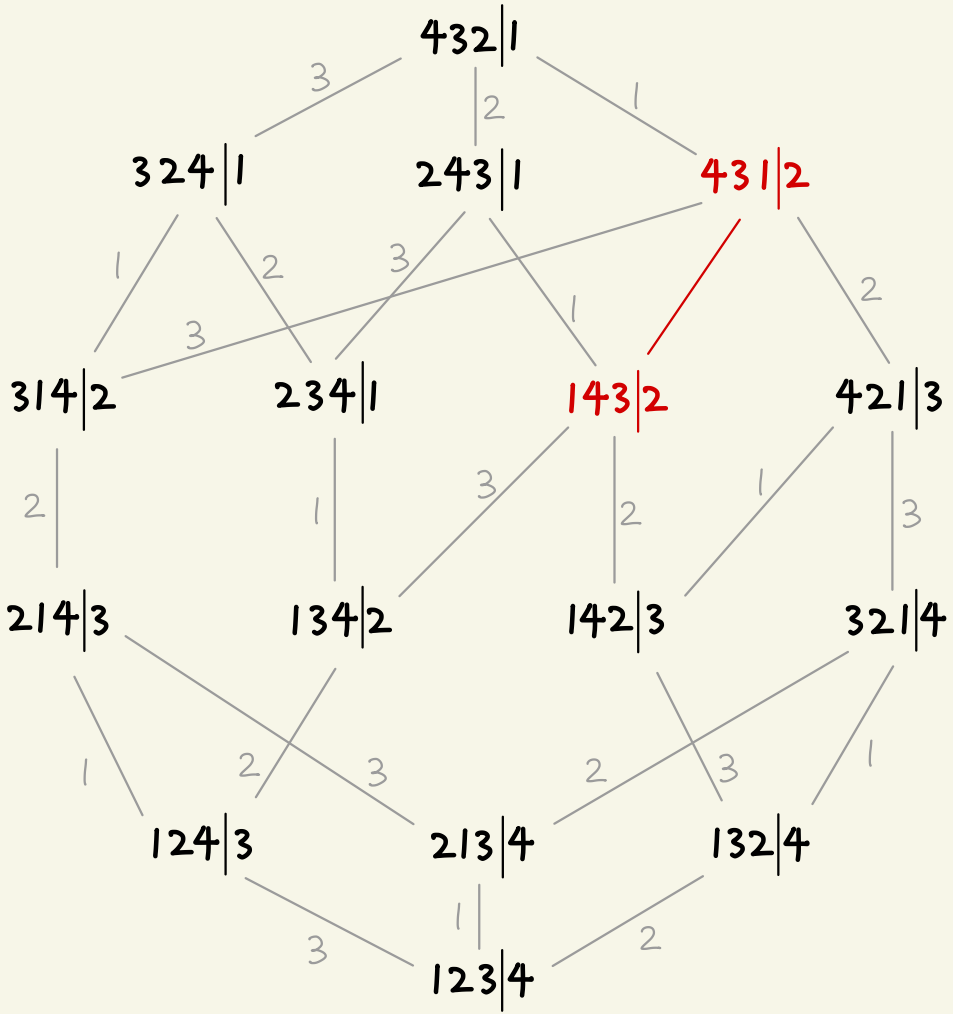
$\pi = 143|2 \quad \tau = 431|2$

$$R(\tau) = \left\{ \begin{matrix} 2132 & 2312 \\ 1232 & 1323 \\ 3123 \end{matrix} \right\}$$

$$R(\pi) = \{ 232 \quad 323 \}$$

$$\ell(\pi)+1 = \ell(\tau)$$

$$\Rightarrow \pi \lessdot^{\mu} \tau$$



Ex.

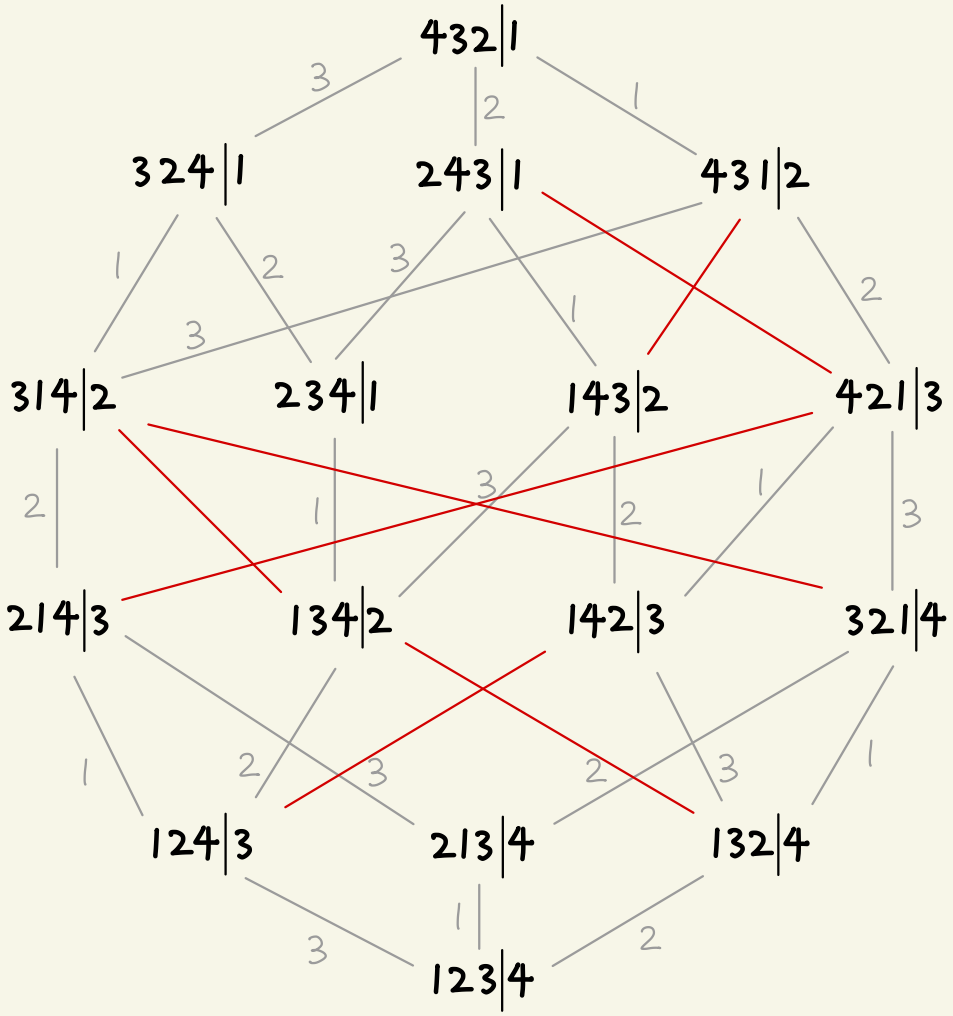
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$$R(\pi) = \{ 232 \quad 323 \}$$

$$\ell(\pi)+1 = \ell(\tau)$$

$$\Rightarrow \pi \lessdot^{\mu} \tau$$



Strong Bruhat Order

Def: Operator t_{ij}^{μ} on $\pi \in I_\mu$ $(i<j)$

- Swap values i and j in $Cyc(\pi)$
- Break $(a,b) \to (a)(b)$ if $a<b$
- Two special cases when i, j in the same block with $i<k<\ell<j$

$$(k,i)(j,\ell) \longrightarrow (j,i)(k)(\ell)$$

$$(\ell,i)(j,k) \longrightarrow (j,i)(\ell,k)$$

Ex.

$$t_{25}^{\mu}(2|176|543)$$

$$(2)|(1)(7,6)|(5,3)(4)$$

$$(5)|(1)(7,6)|(2,3)(4)$$

$$(5)|(1)(7,6)|(2)(3)(4)$$

$$= 5|176|234$$

Ex.

$t^{\mu}_{25}(2|176|543)$

$(2)|(1)(7,6)|(5,3)(4)$

$(5)|(1)(7,6)|(2,3)(4)$

$(5)|(1)(7,6)|(2)(3)(4)$

$= 5|176|234$

$t^{\mu}_{16}(5|4613|72)$

$(5)|(4,1)(6,3)|(7,2)$

$(5)|(6,1)(4,3)|(7,2)$

$= 5|6431|72$

Ex.

$t^{\mu}_{25}(2|176|543)$

$(2)|(1)(7,6)|(5,3)(4)$

$(5)|(1)(7,6)|(2,3)(4)$

$(5)|(1)(7,6)|(2)(3)(4)$

$= 5|176|234$

$t^{\mu}_{16}(5|4613|72)$

$(5)|(4,1)(6,3)|(7,2)$

$(5)|(6,1)(4,3)|(7,2)$

$= 5|6431|72$

$= t^{\mu}_{13}(5|4613|72)$

Strong Bruhat Order

Thm [C. - Hamaker] For $\pi, \tau \in I_\mu$,

$$\pi \lessdot^\mu \tau \quad \text{iff} \quad \begin{array}{l} t_{ij}^\mu(\pi) = \tau \quad \text{for some } i, j \\ \text{and} \quad \ell(\pi)+1 = \ell(\tau) \end{array}$$

Strong Bruhat Order

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Ex. $\pi = 2|176|543$, $\tau = 5|176|234$, $t_{25}^\mu(\pi) = \tau$

$\ell(\pi) = 10$, $\ell(\tau) = 11 \implies \pi \lessdot^\mu \tau$

Ex. $\pi = 5|4613|72$, $\tau = 5|6431|72$, $t_{16}^\mu(\pi) = \tau$

$\ell(\pi) = 11$, $\ell(\tau) = 12 \implies \pi \lessdot^\mu \tau$

Strong Bruhat Order

Thm [C.-Hamaker] For $\pi <^{\mu} \tau$,

$$A_{\mu}^{-1}(\tau) = \left\{ t_{ij} w : w \in A_{\mu}^{-1}(\pi),\ w <: t_{ij} w,\ t_{ij}^{\mu}(\pi) = \tau \right\}$$

Ex. $\pi = 2|176|543 \xrightarrow{t_{25}^{\mu}} 5|176|234 = \tau$

$A_{\mu}^{-1}(\pi) =$

$$2176453 \xrightarrow{t_{25}} \cancel{5176423}$$

$$2176534 \xrightarrow{t_{25}} 5176234 = A_{\mu}^{-1}(\tau)$$

Schubert Calculus on μ-Involutions

Background

$$Fl(n) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim(V_i) = i \right\}$$

$$G = GL_n(\mathbb{C}) \curvearrowright Fl(n)$$

Background

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$G = GL_n(\mathbb{C}) \curvearrowright Fl(n)$, $H \subseteq G$ action with finite orbits

Borel

H_μ

$O(n)$

Background

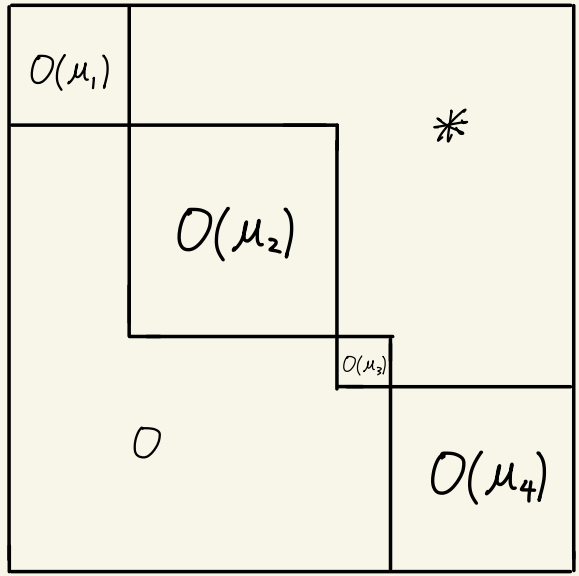
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	Orbits	Cohomology
Borel	Permutations S_n	Schubert polynomials $\mathfrak{S}_w$
H_μ		
$O(n)$		

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Borel	Permutations S_n	Schubert polynomials $\mathfrak{S}_w$
H_μ		
$O(n)$	Involutions I_n	Involution Schubert polynomials $\mathfrak{S}_z^I$

Background

$$Fl(n) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim(V_i) = i \right\}$$

$G = GL_n(\mathbb{C}) \curvearrowright Fl(n)$, $H \subseteq G$ action with finite orbits

	Orbits	Cohomology
Borel	Permutations S_n	Schubert polynomials $\mathfrak{S}_w$
H_μ	μ-Involutions I_μ	μ-Involution Schubert polynomials $\mathfrak{S}_\pi^\mu$
$O(n)$	Involutions I_n	Involution Schubert polynomials $\mathfrak{S}_z^I$

Background

The Variety of Complete Quadrics

$$= \text{wonderful compactification of } {}^{GL(n)}/_{O(n)}$$

Borel-orbits of the Variety of Complete Quadrics

$$= H_u\text{-orbits of the Flag Variety}$$

Def: For $\pi \in I_\mu$, its <u>μ-involution Schubert polynomial</u> is

$$\mathfrak{S}_\pi^\mu := \sum_{w \in A_\mu(\pi)} \mathfrak{S}_w$$

Ex. $\pi = 421|3$ $\qquad A_\mu^{-1}(\pi) = \{4123,\ 2413\}$

$$\mathfrak{S}_\pi^\mu = \mathfrak{S}_{2341} + \mathfrak{S}_{3142}$$

$$= x_1 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3$$

Thm [C.-Hamaker] If ν refines μ, then we can expand any $\mathfrak{G}^{\mu}_{\pi}$ as a multiplicity-free positive sum of ν-Involution Schubert polynomials.

Ex. $\mu = (4,2)$ $\nu = (2,2,2)$

$$\mathfrak{G}^{\mu}_{6431|52} = \mathfrak{G}^{\nu}_{43|61|52} + \mathfrak{G}^{\nu}_{61|43|52} + \mathfrak{G}^{\nu}_{46|13|52}$$

Transition

Thm [C.-Hamaker] For $\pi \in I_\mu$ and $(j,i) \in Cyc(\pi)$,

$$2^{-\delta(i,j)} \cdot (X_i + X_j)\mathcal{G}_\pi^\mu = \sum_{\tau \in \Phi} \mathcal{G}_\tau^\mu - \sum_{\tau \in \Psi} \mathcal{G}_\tau^\mu$$

$$\Phi = \left\{ \tau = t_{ab}^\mu(\pi) : \pi \lessdot^\mu \tau \,,\ a \in \{i,j\} \right\}$$

$$\Psi = \left\{ \tau = t_{ab}^\mu(\pi) : \pi \lessdot^\mu \tau \,,\ b \in \{i,j\} \right\}$$

Transition $\pi = 34|521|6$ $(3)(4)|(5,1)(2)|(6)$

Bruhat covers: t_{16}^{μ} t_{26}^{μ} t_{34}^{μ} t_{45}^{μ} t_{56}^{μ}

Transition $\pi = 34|521|6$ $(3)(4)|(5,1)(2)|(6)$

Bruhat covers: t^u_{16} t^u_{26} t^u_{34} t^u_{45} t^u_{56}

$$X_4 \cdot \mathcal{G}^u_\pi = \mathcal{G}^u_{t^u_{45}(\pi)} - \mathcal{G}^u_{t^u_{34}(\pi)}$$

Transition $\pi = 34|521|6$ $\quad (3)(4)|(5,1)(2)|(6)$

Bruhat covers: $t_{16}^{\mu}\ t_{26}^{\mu}\ t_{34}^{\mu}\ t_{45}^{\mu}\ t_{56}^{\mu}$

$$X_4 \cdot \mathcal{G}_{\pi}^{\mu} = \mathcal{G}_{t_{45}^{\mu}(\pi)}^{\mu} - \mathcal{G}_{t_{34}^{\mu}(\pi)}^{\mu}$$

$$(X_1 + X_5) \cdot \mathcal{G}_{\pi}^{\mu} = \mathcal{G}_{t_{16}^{\mu}(\pi)}^{\mu} + \mathcal{G}_{t_{56}^{\mu}(\pi)}^{\mu} - \mathcal{G}_{t_{45}^{\mu}(\pi)}^{\mu}$$

Transition $\pi = 34|521|6$ $\quad (3)(4)|(5,1)(2)|(6)$

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RMK: $X_1 \cdot \mathcal{G}_{\pi}^{\mu} \notin \mathbb{Z}[\mathcal{G}_{\tau}^{\mu}]$

Thm For $\pi \in I_\mu$ with blocks $B_1 \cdots B_k$,

$$A_\mu^{-1}(\pi) = \left\{ Q_1 Q_2 \cdots Q_k : Q_i \in A^{-1}(B_i) \right\}$$

Thm For $\pi, \tau \in I_\mu$, $\pi \lessdot^\mu \tau$ iff

$t_{ij}^\mu(\pi) = \tau$ for some i, j

and $\ell(\pi)+1 = \ell(\tau)$

Thm For $\pi \lessdot^\mu \tau$,

$$A_\mu^{-1}(\tau) = \left\{ t_{ij} w : w \in A_\mu^{-1}(\pi),\ w \lessdot t_{ij} w,\ t_{ij}(\pi) = \tau \right\}$$

Thm For $\pi \in I_\mu$ and $(j,i) \in Cyc(\pi)$,

$$2^{-\delta(i,j)} \cdot (x_i + x_j) G_\pi^\mu = \sum_{\tau \in \Phi} G_\tau^\mu - \sum_{\tau \in \Psi} G_\tau^\mu$$

Thank You !!

Merci !!