μ-Involution Schubert Polynomials

Jack Chou 周辰恩
joint with Zach Hamaker
University of Florida

BICMR, Peking University

Outline

Permutations (μ-involutions)

Variety of Complete Quadrics

Coxeter Combinatorics for μ-Involutions

Transition Equation

Background

Involution : 625431 $(w^{-1} = w)$

Cycle notation : $(6,1)(5,3)(2)(4)$

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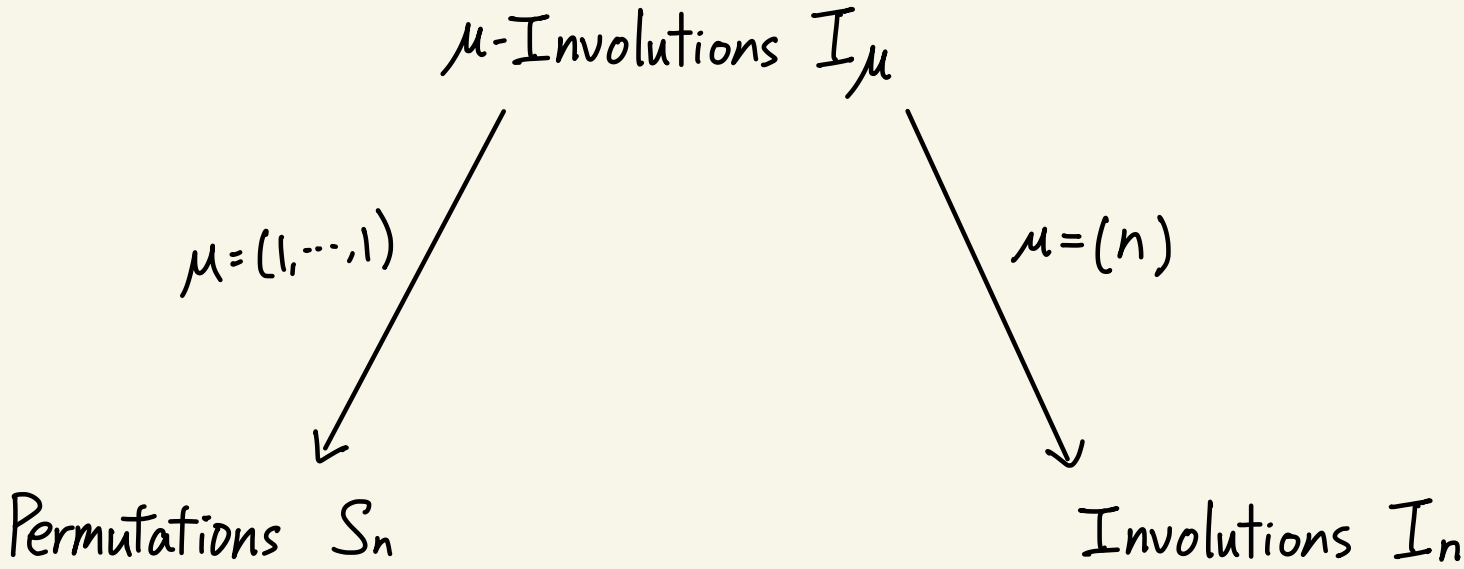
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μ-Cycle notation: $(4)\,|\,(5,1)(3)(7)\,|\,(8)\,|\,(9,2)(6)$

Background


μ-Involutions I_μ
$\mu=(1,\cdots,1)$
$\mu=(n)$
Permutations S_n
Involutions I_n

Background

$$Fl(n) = \left\{ 0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n : \dim(V_i) = i \right\}$$

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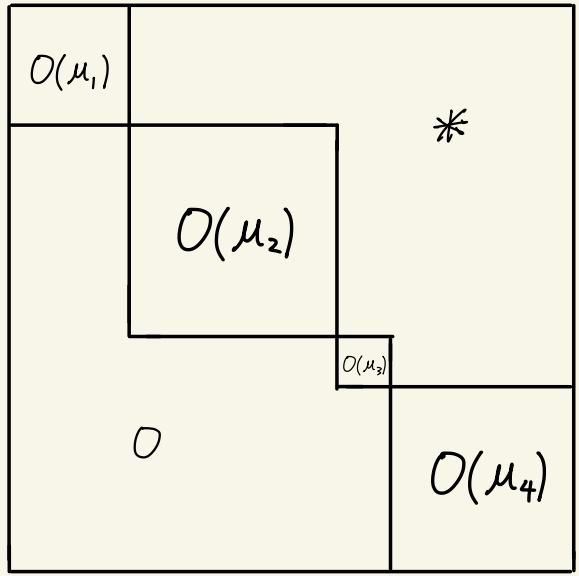
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Borel-orbits of the Variety of Complete Quadrics

= H_u-orbits of the Flag Variety

Coxeter Combinatorics

Weak Order and Atoms

Weak Order

$\mu \vDash n$, I_μ has generators

s_i for $i \in [n-1]$

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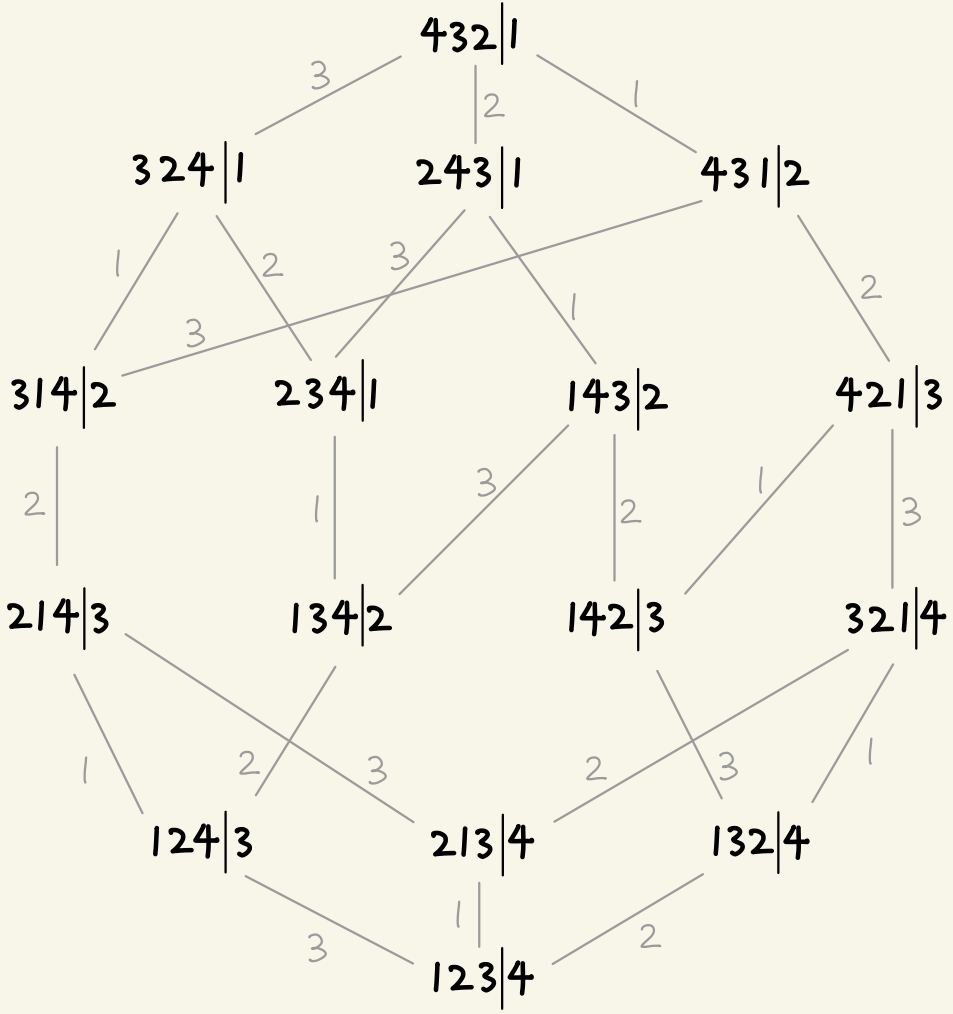
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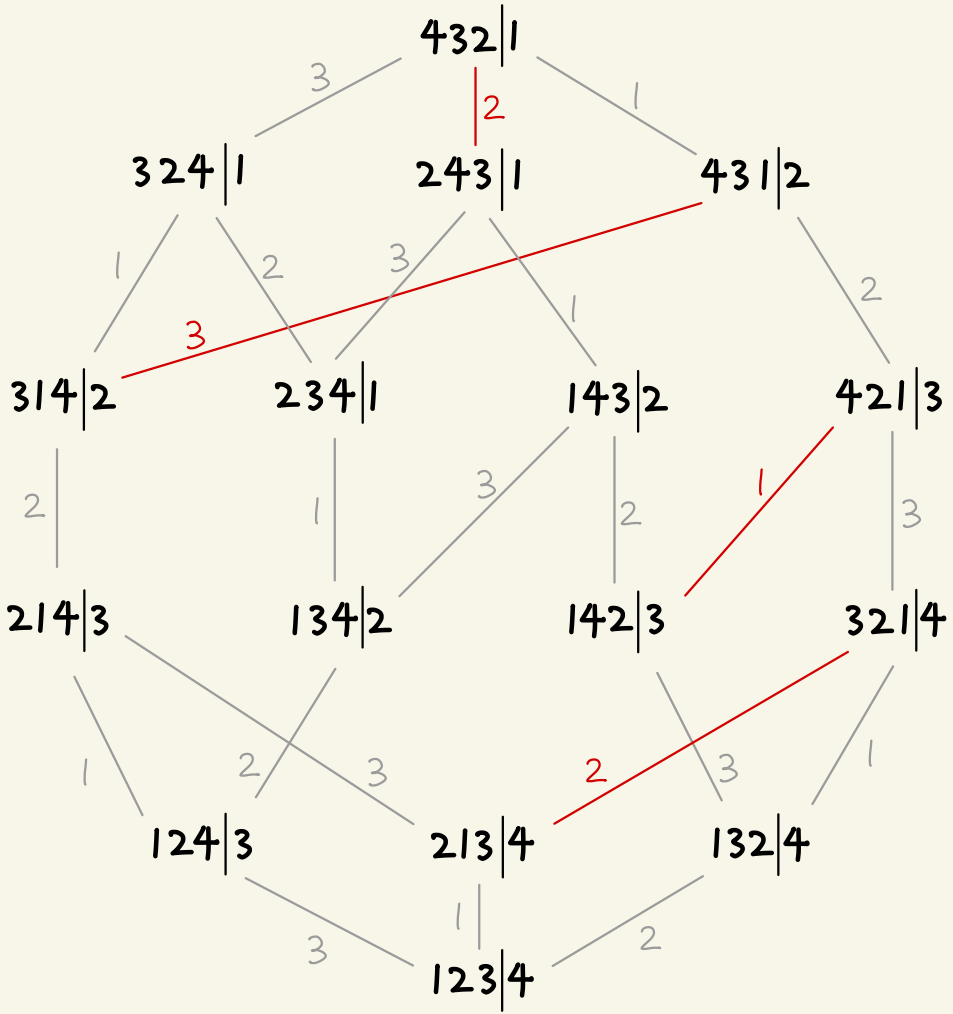
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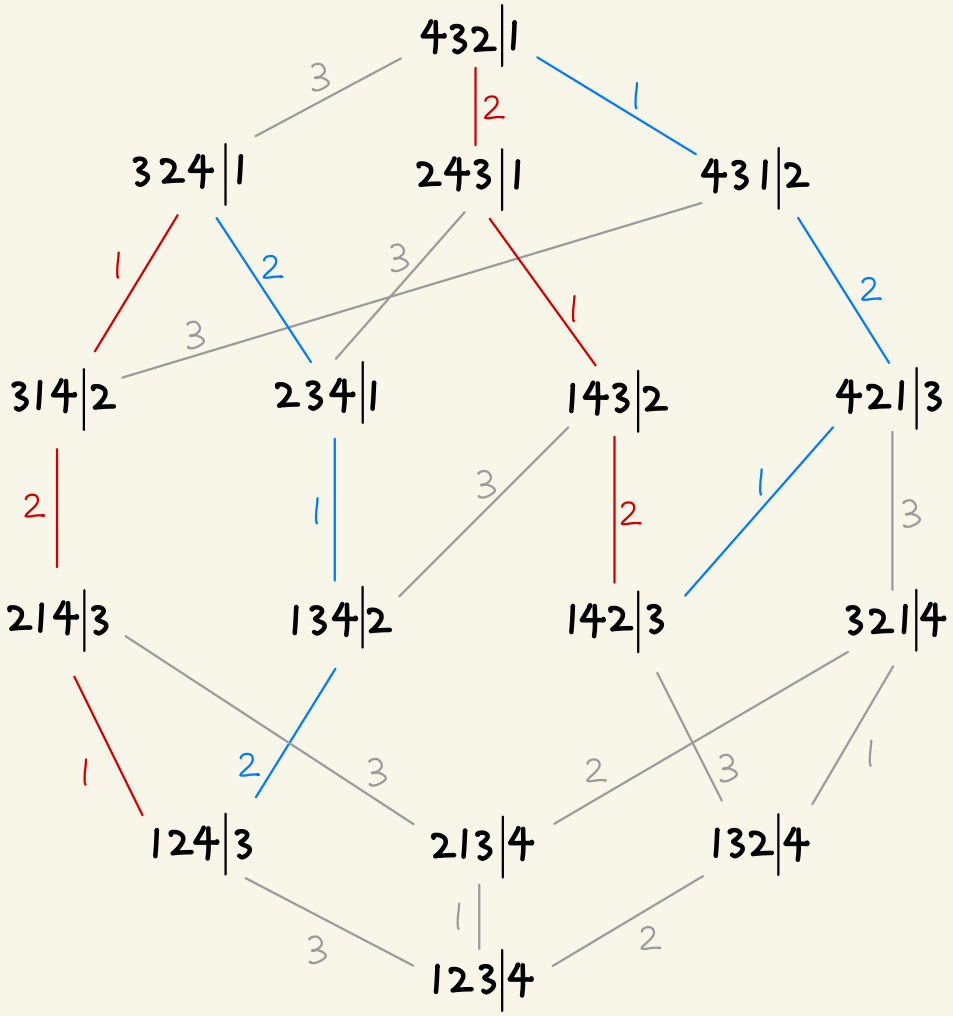
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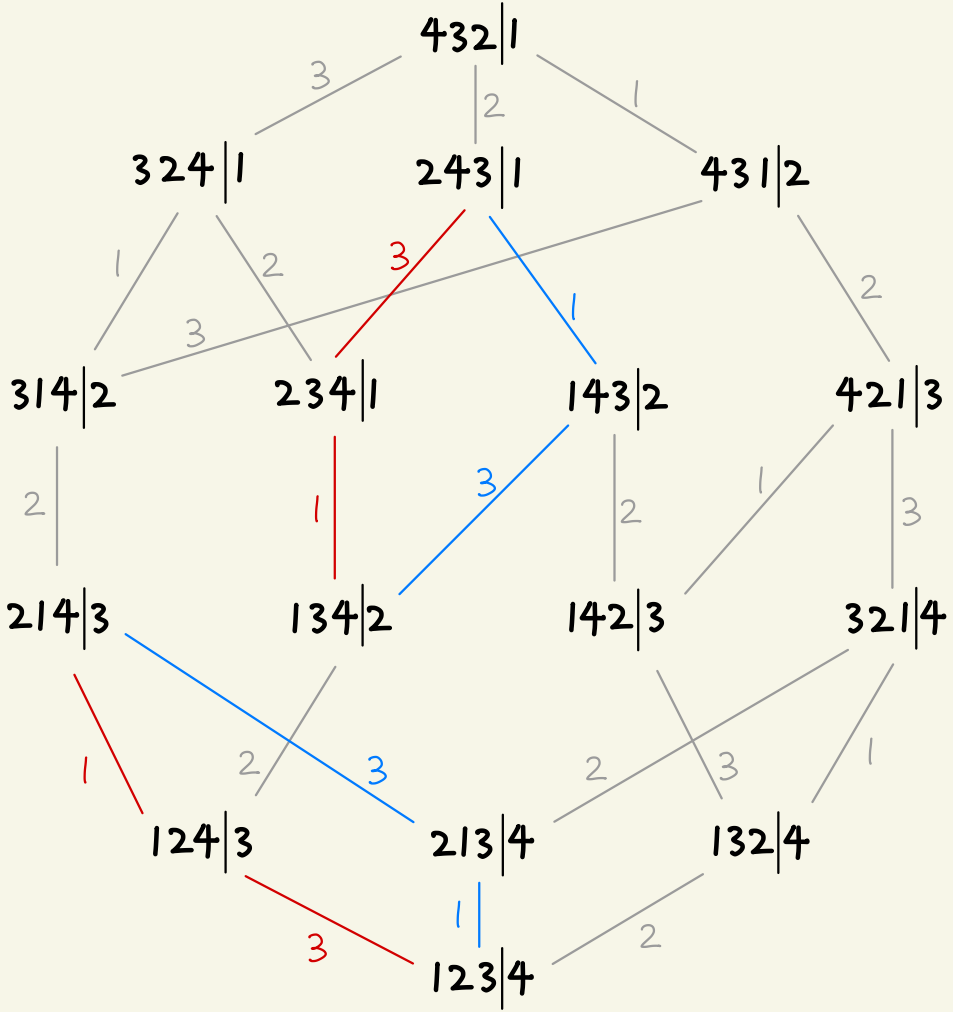
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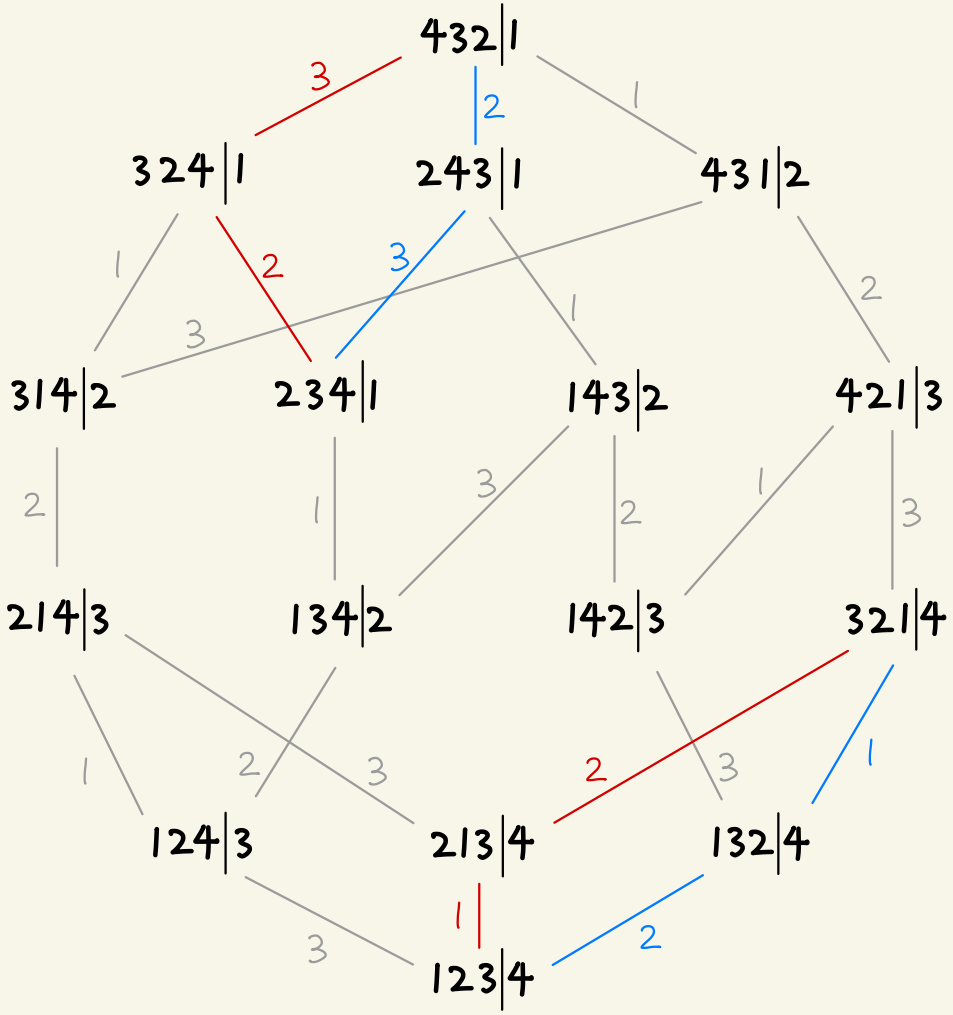
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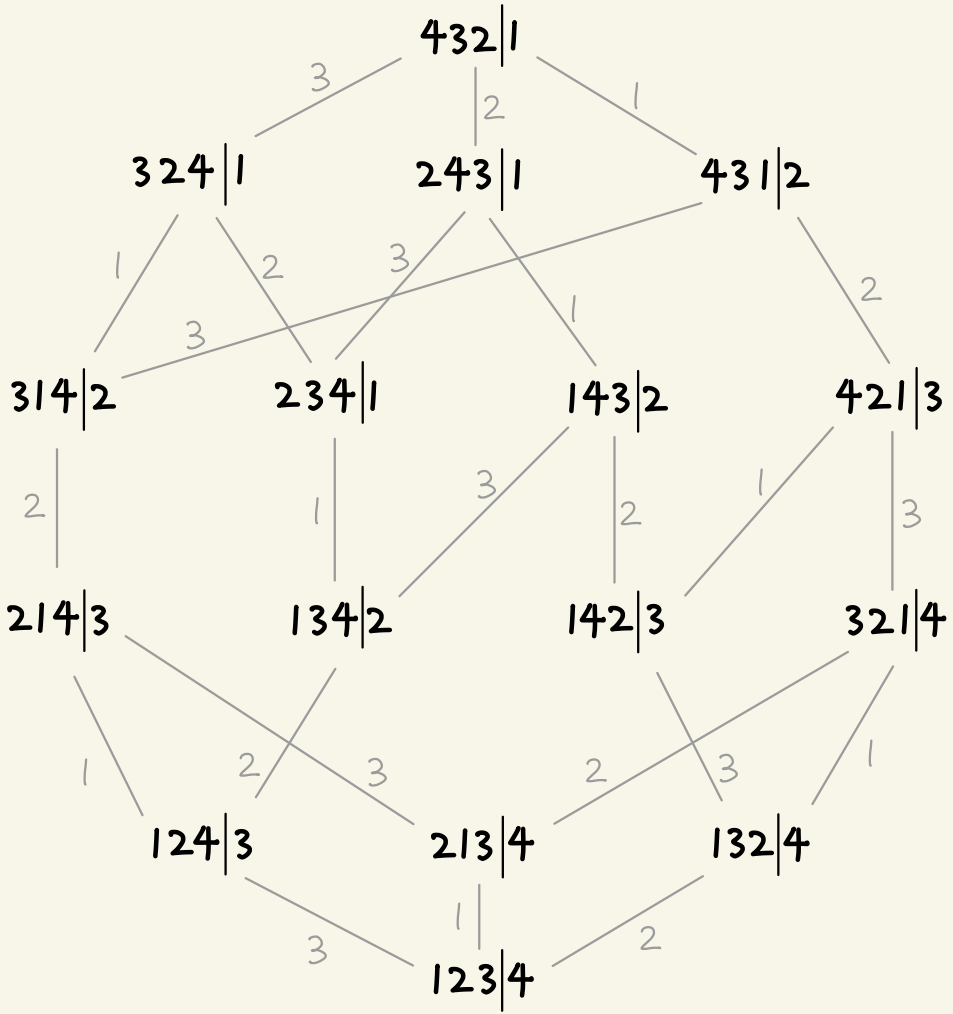
Sometimes

$$\pi \cdot s_i s_{i+1} = \pi \cdot s_{i+1} s_i$$



Words

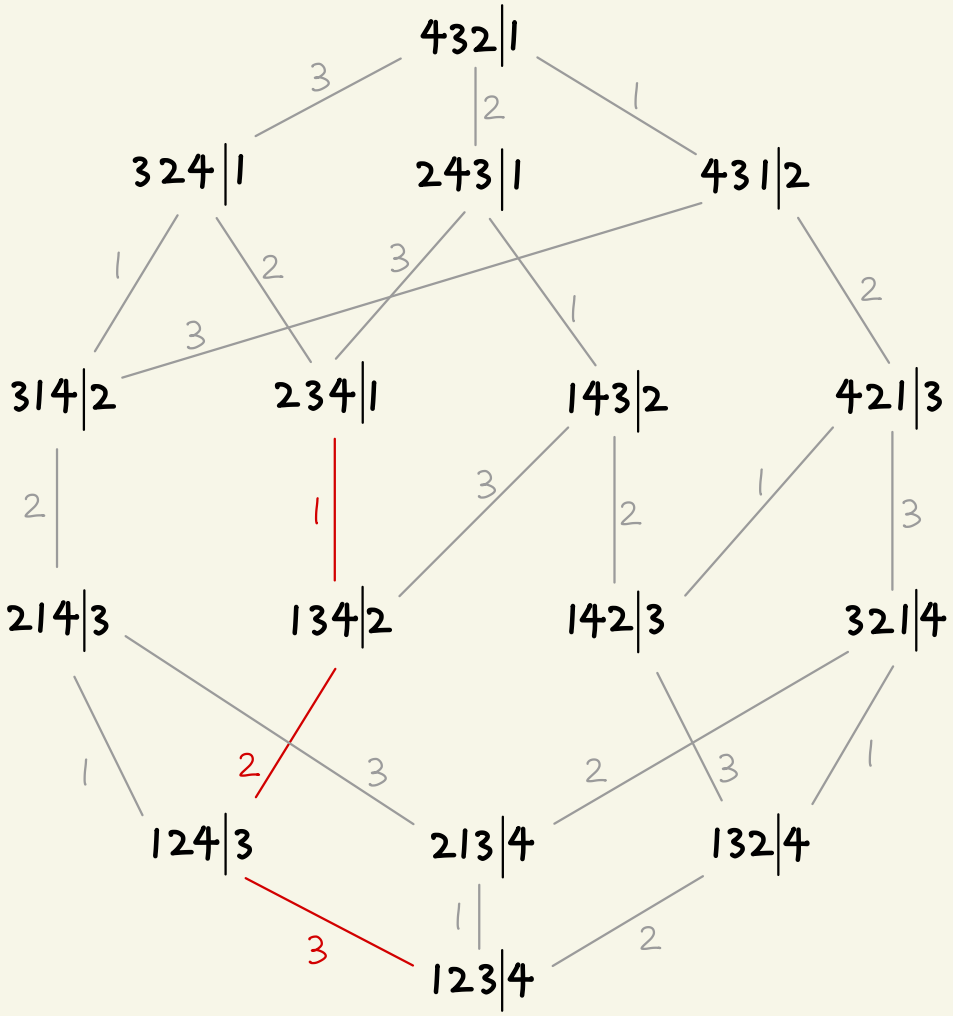
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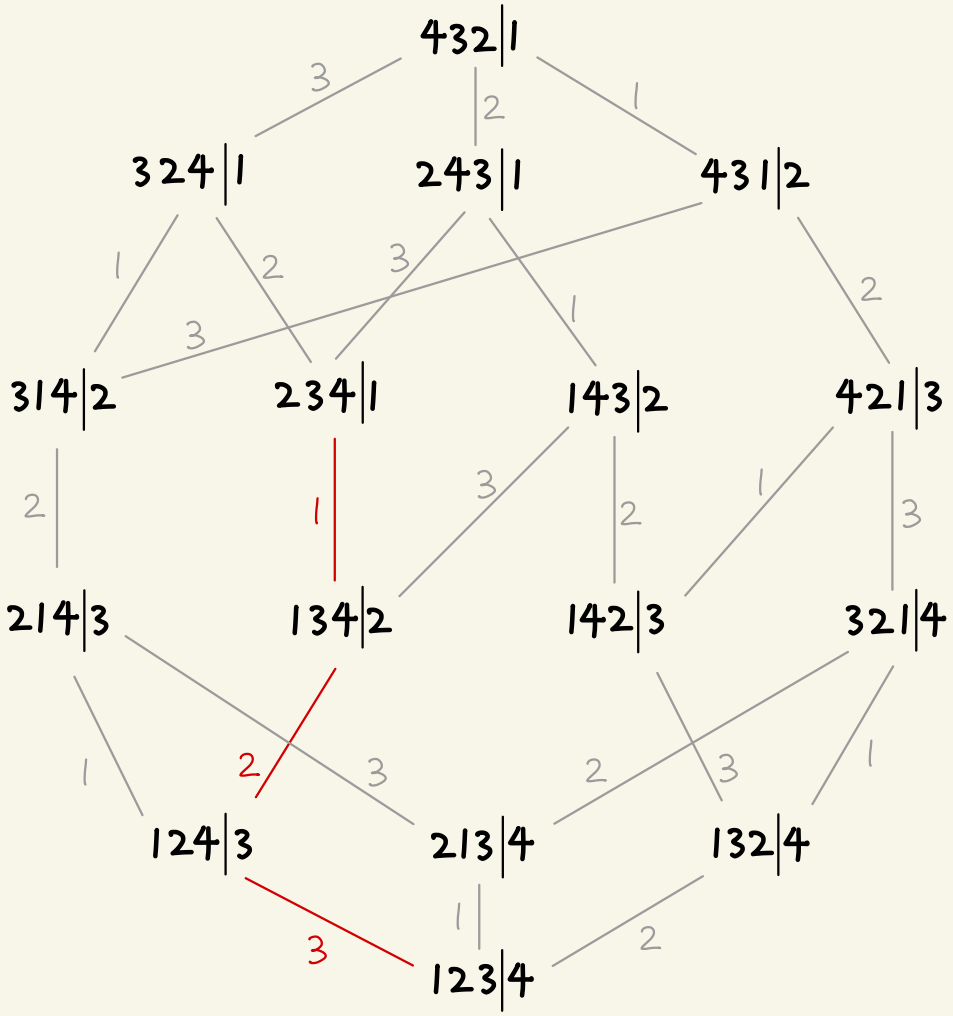
$234|1 = s_3 s_2 s_1$



Words

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$234|1 = s_3 s_2 s_1 = \overline{321}$

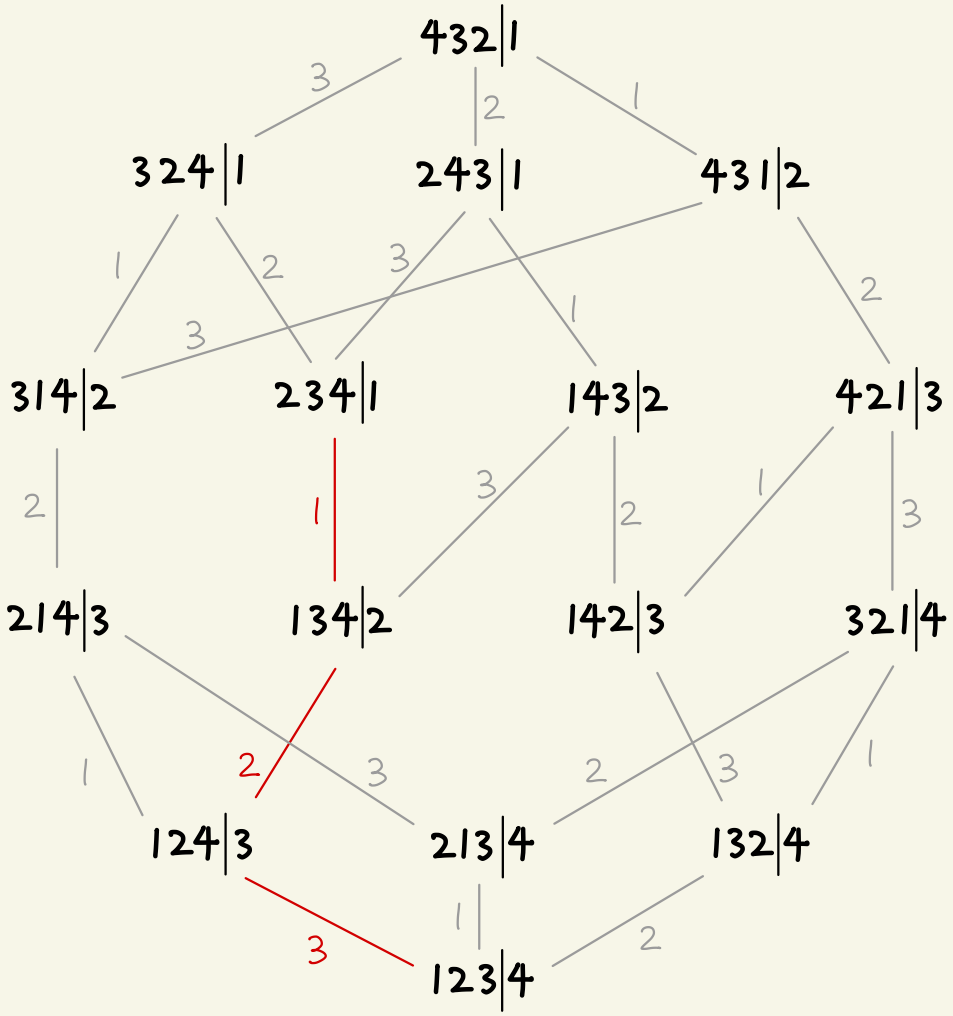


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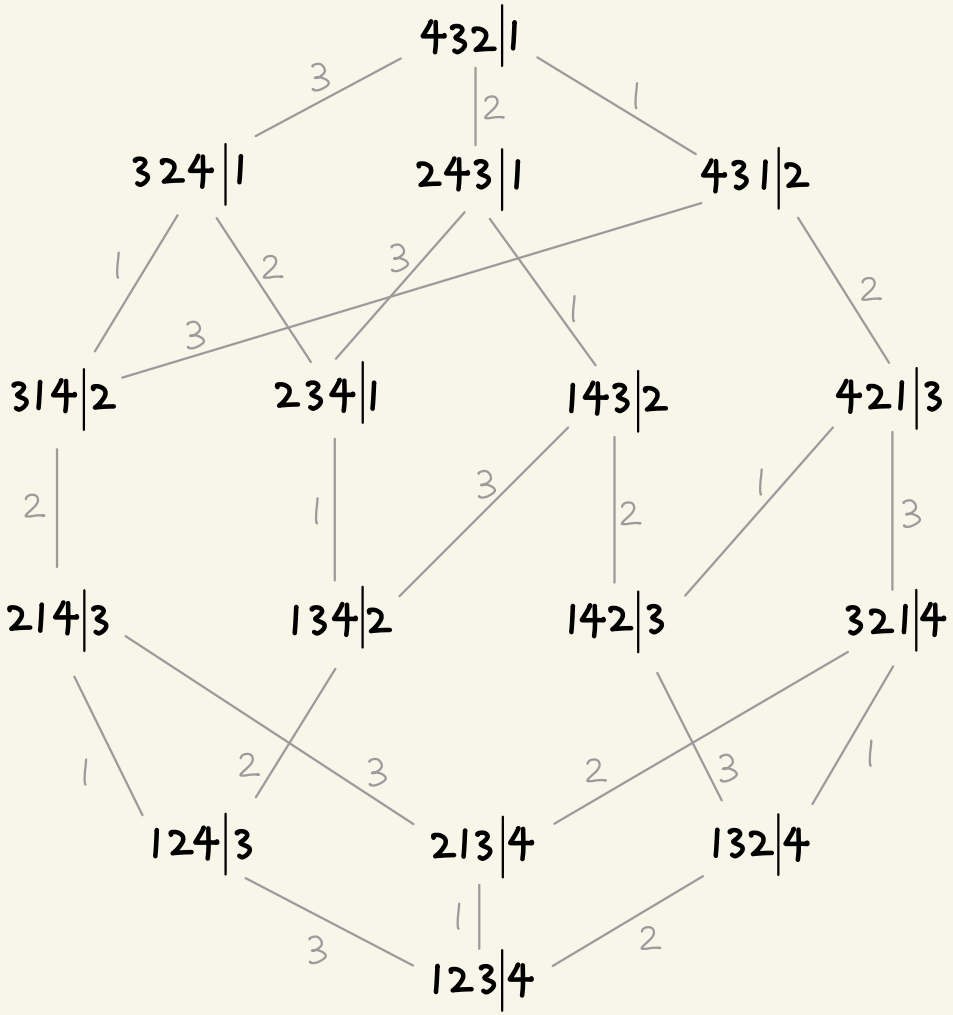
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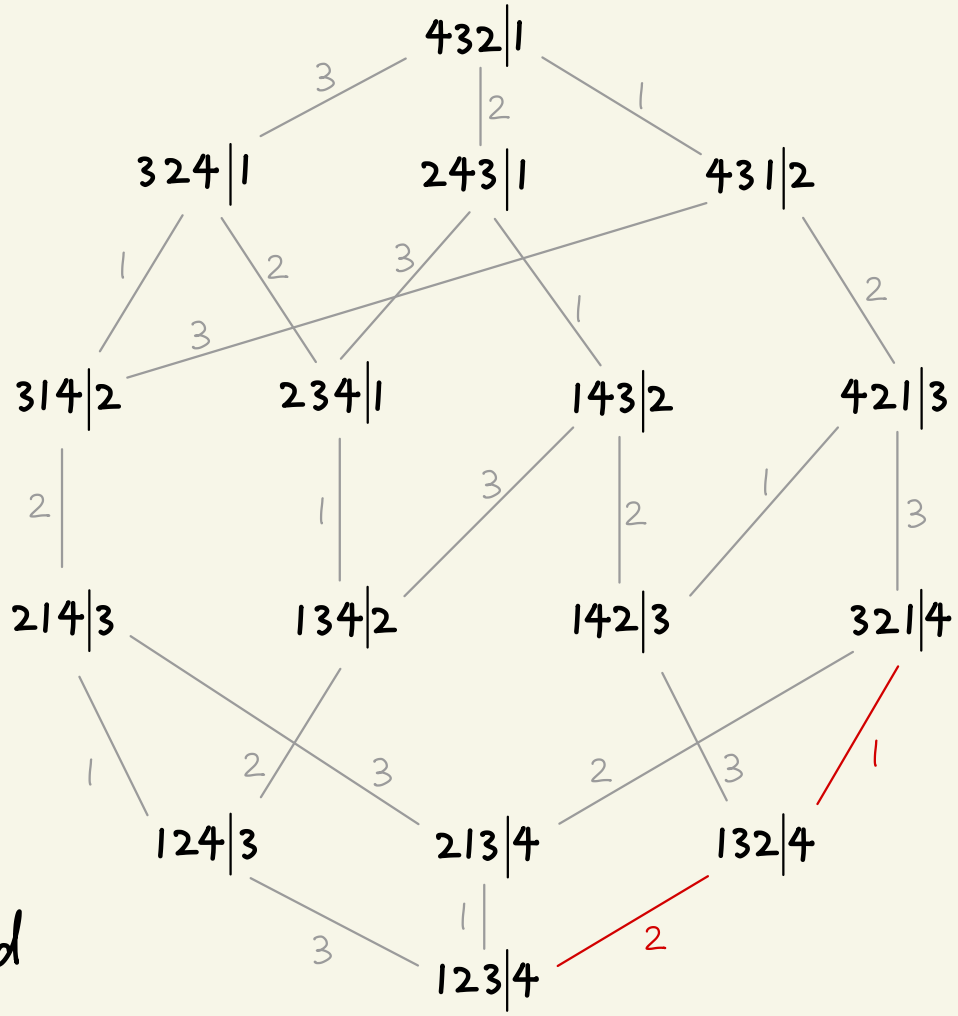
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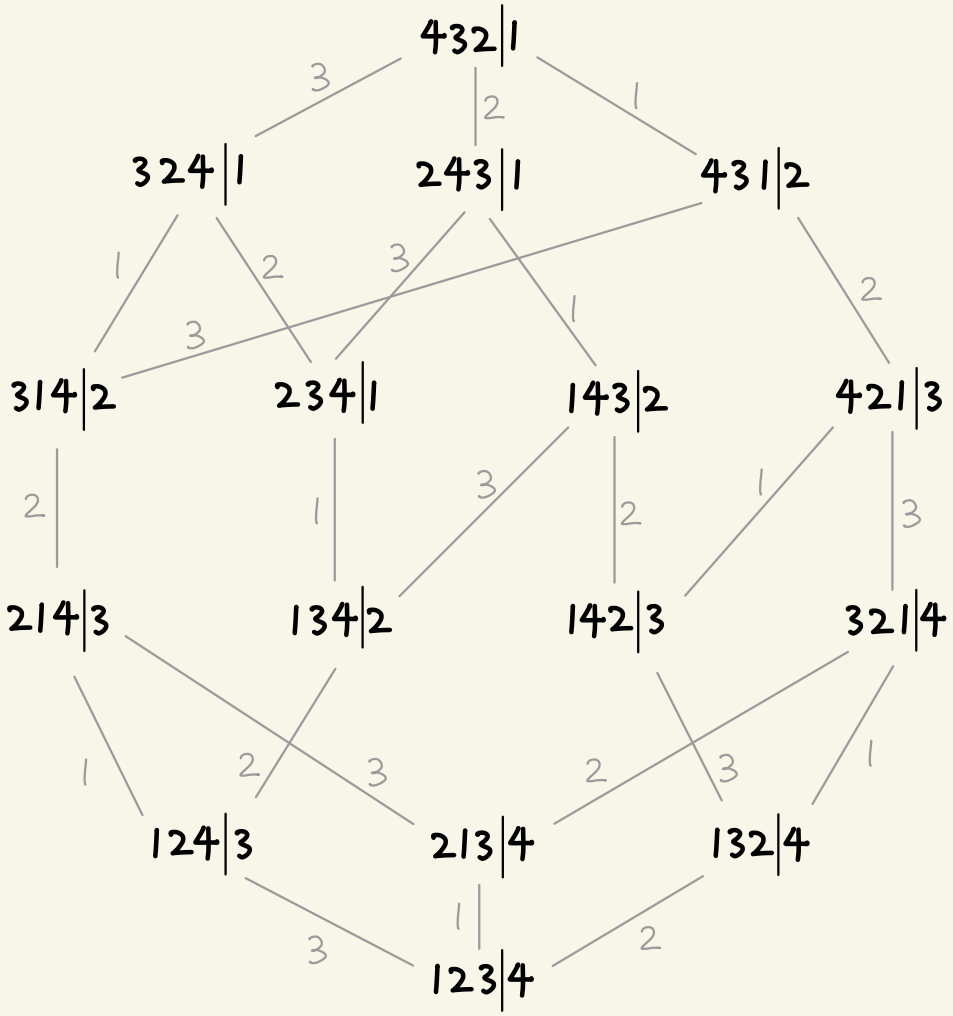
Ex. $\overline{212}$ is not μ-reduced



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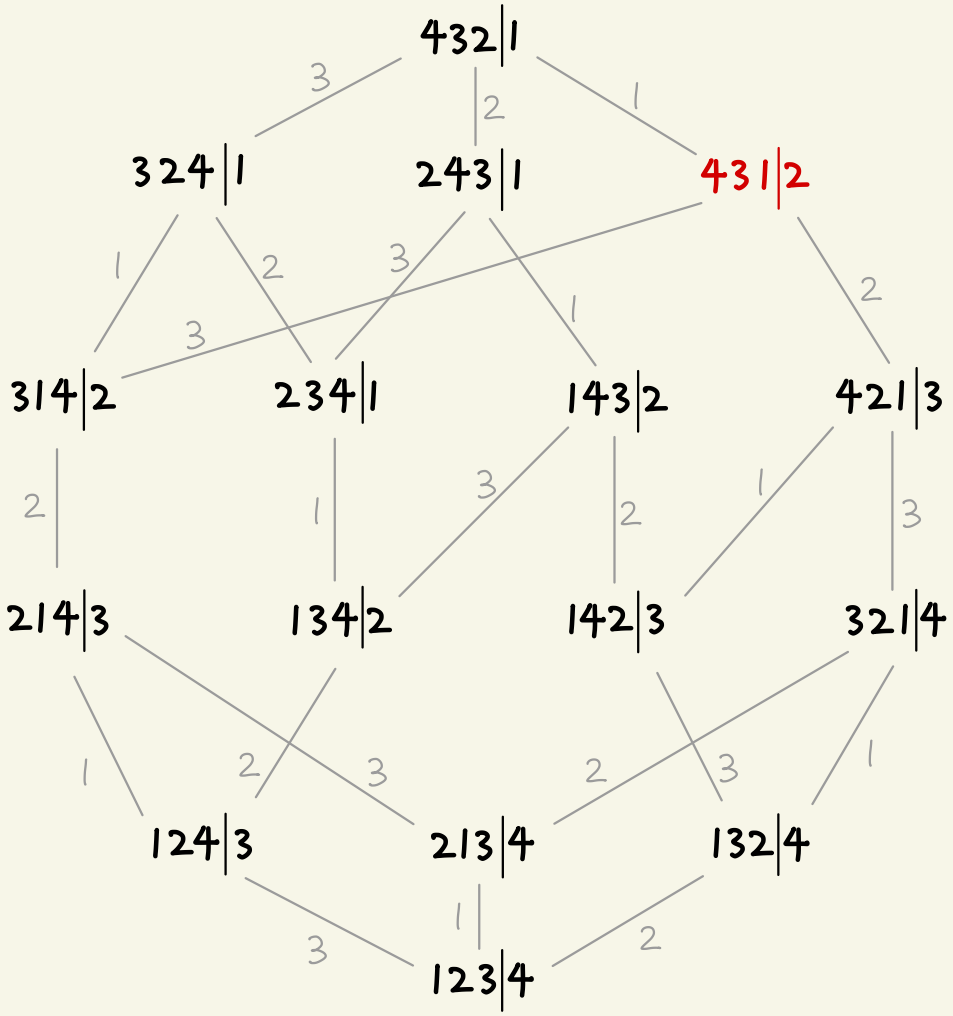
Ex. $\pi = 431|2$

5 length of underlying permutation

2 inversions between blocks

4 n

3 # of cycles



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$\ell(\pi) =$ length of its μ-reduced words

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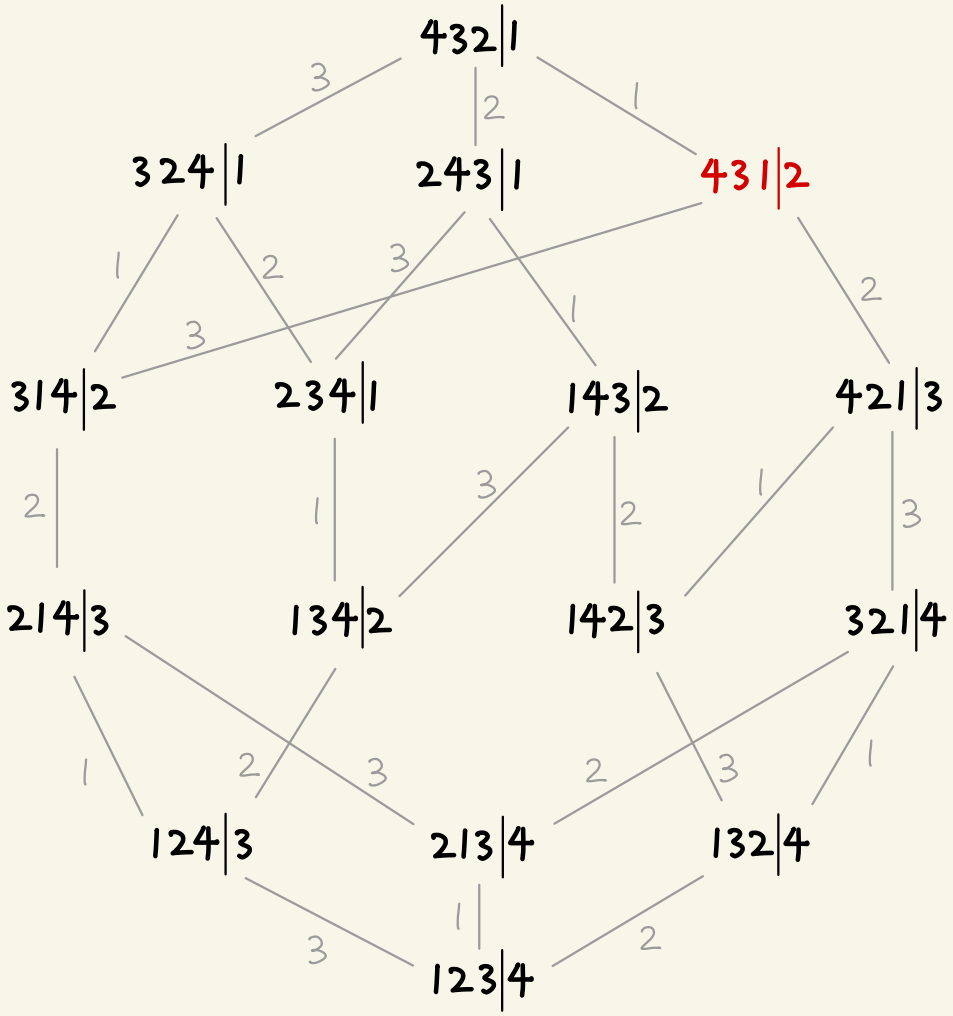
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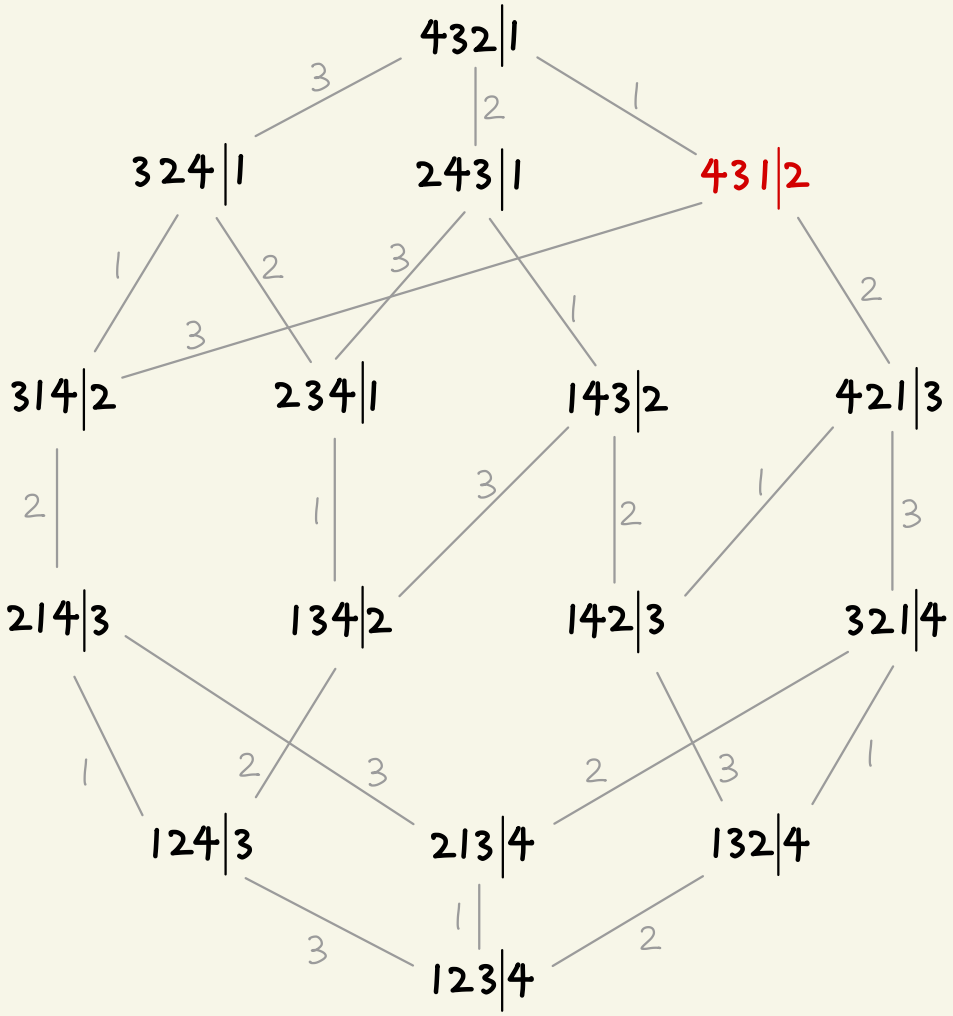
3 # of cycles

$\ell(\pi) = \frac{1}{2}(5+2+4-3) = 4$



Atoms

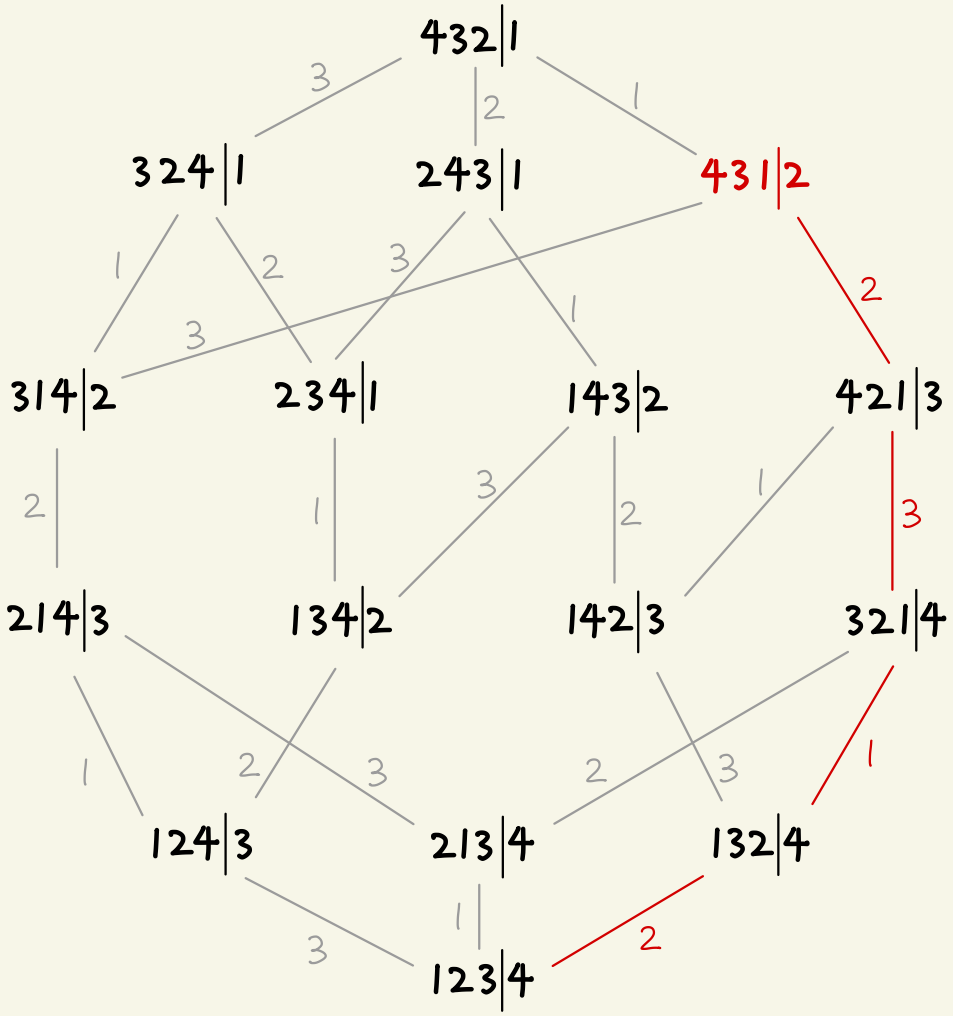
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2132

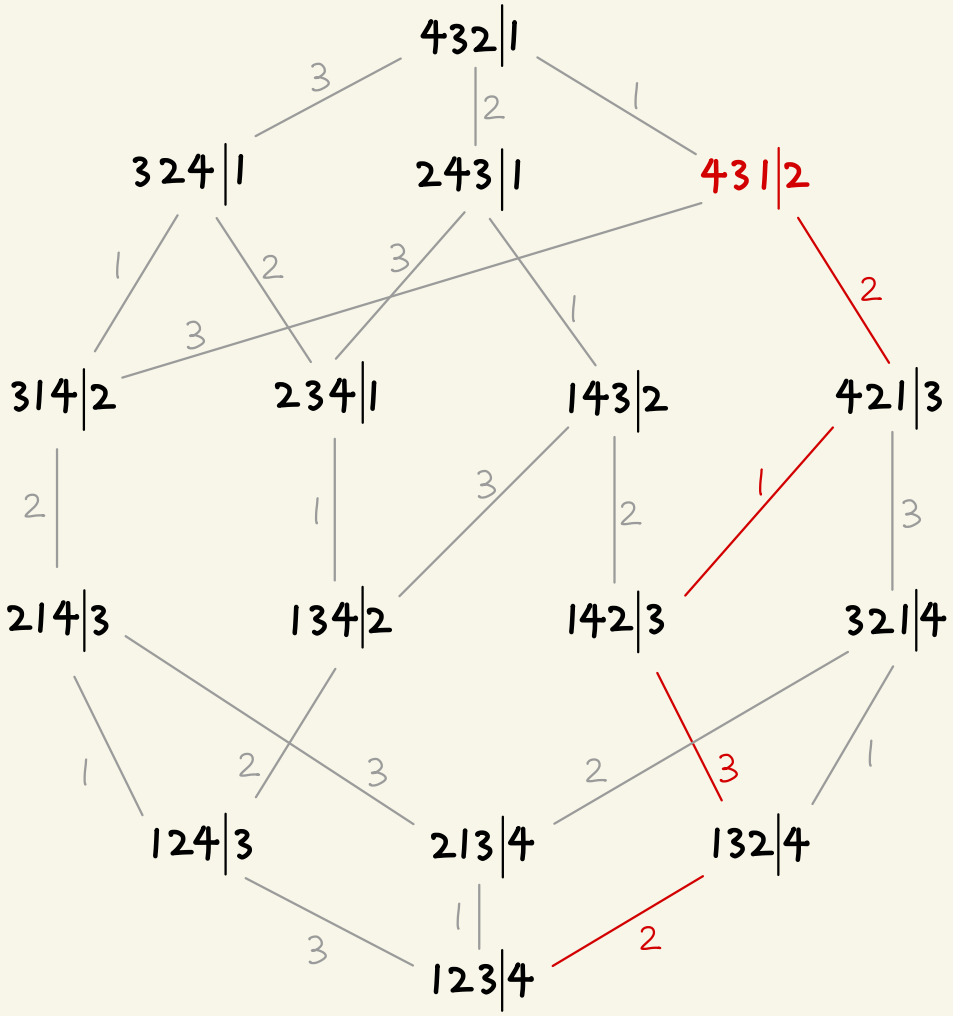


Atoms

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2 1 3 2

2 3 1 2



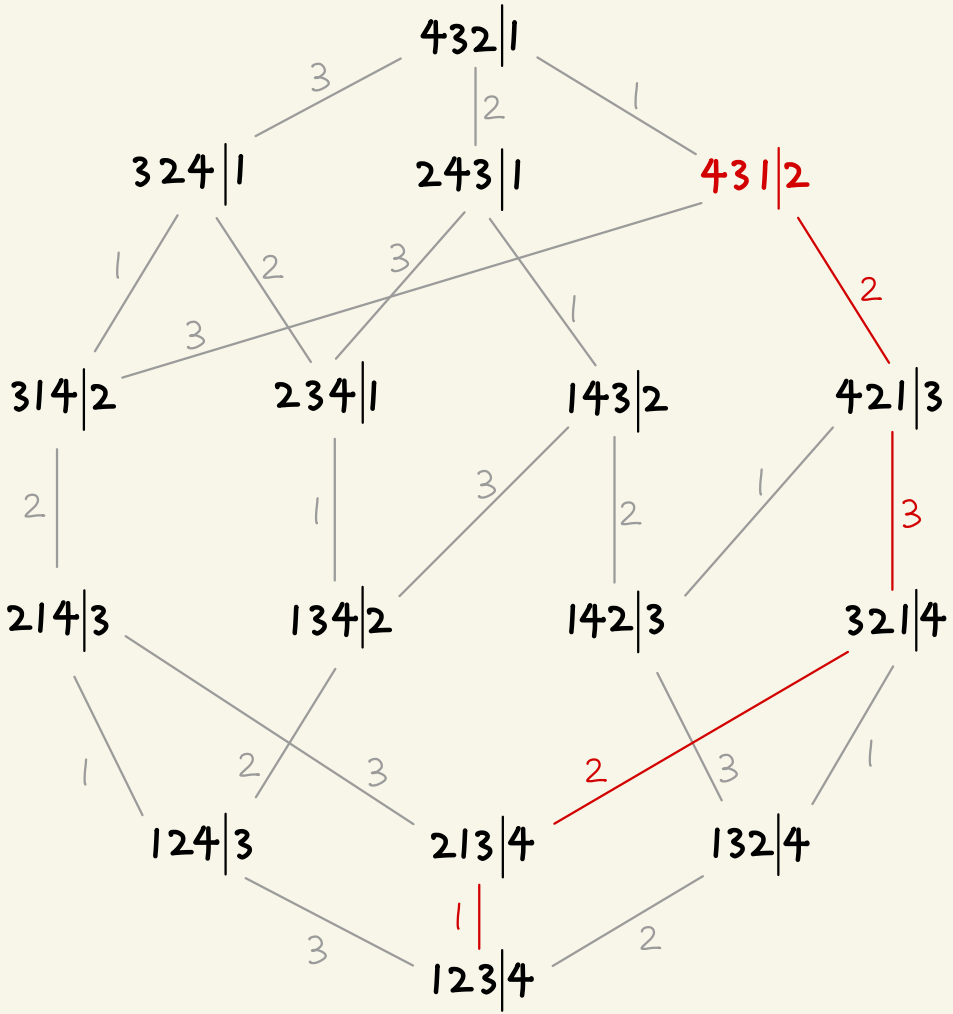
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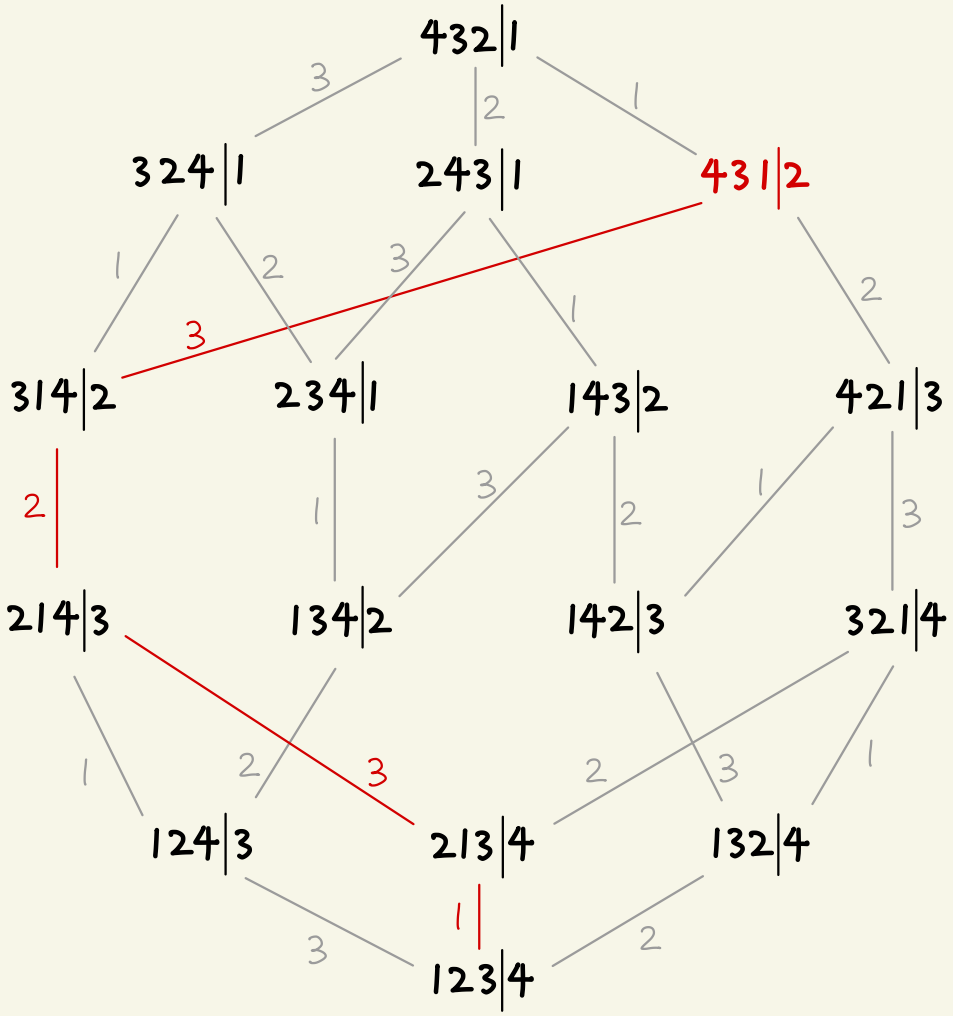
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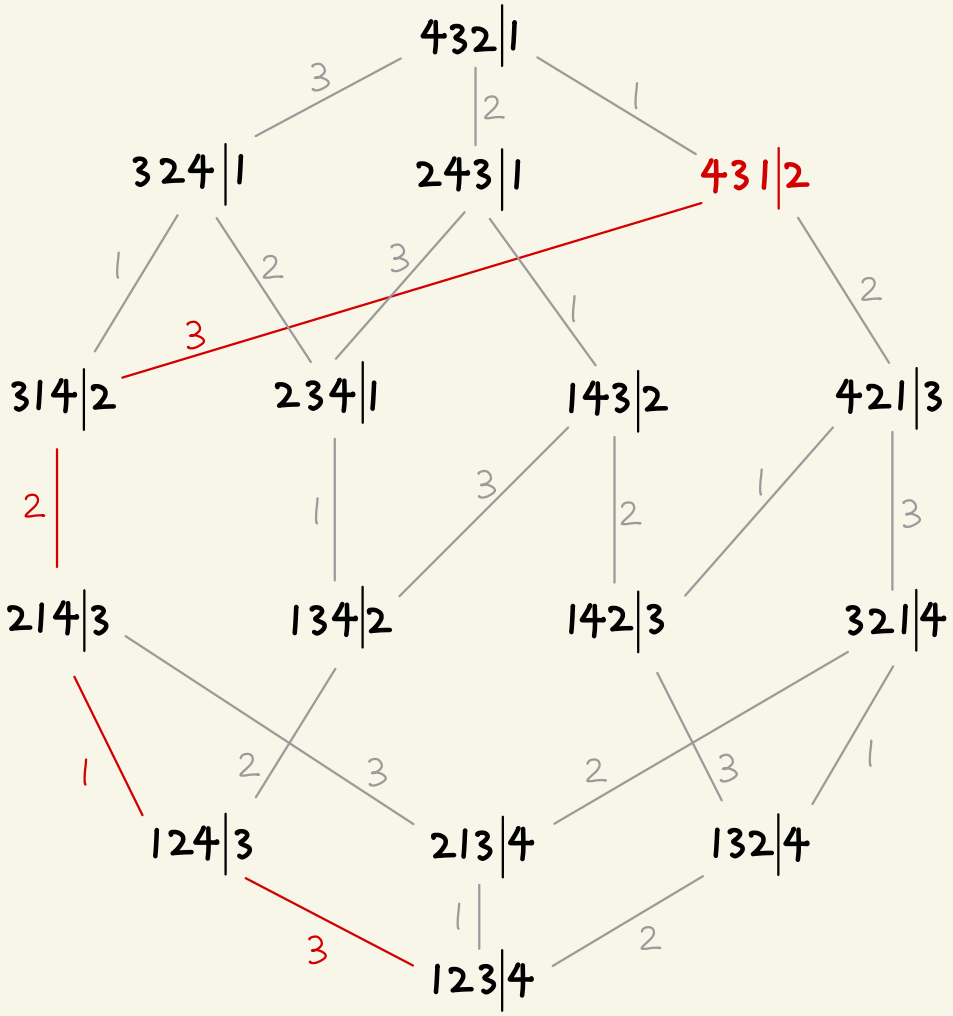
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Atoms

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$\overline{2132}$

$\overline{2312}$

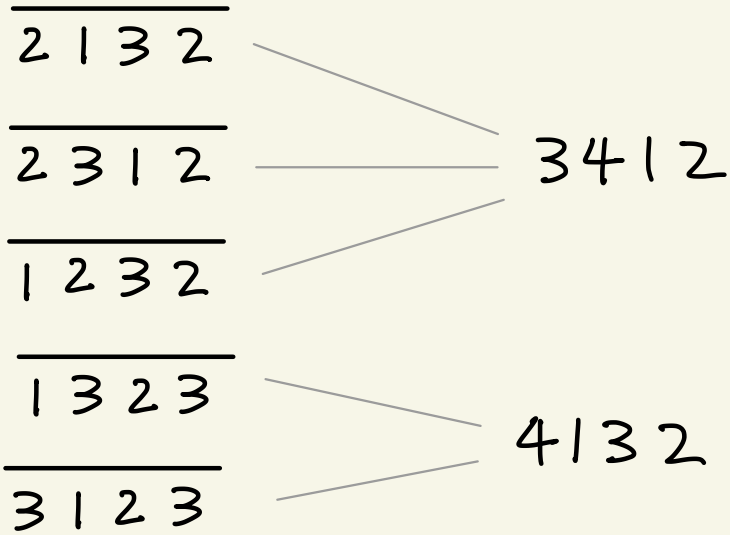
$\overline{1232}$

$\overline{1323}$

$\overline{3123}$

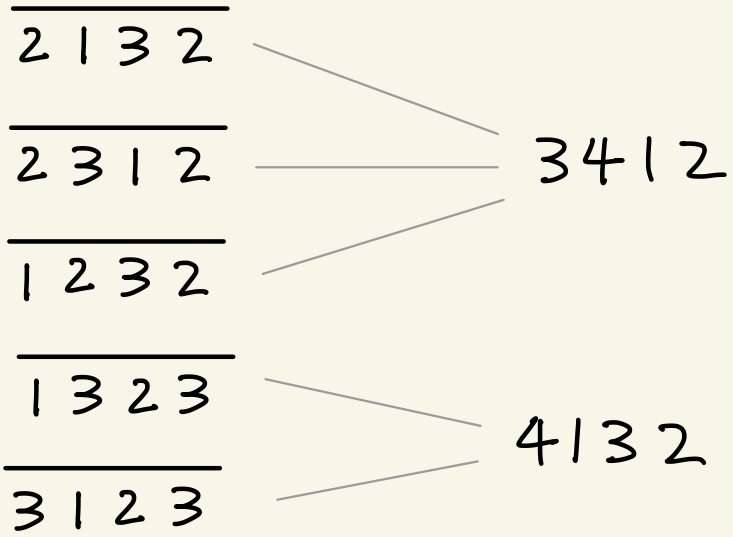
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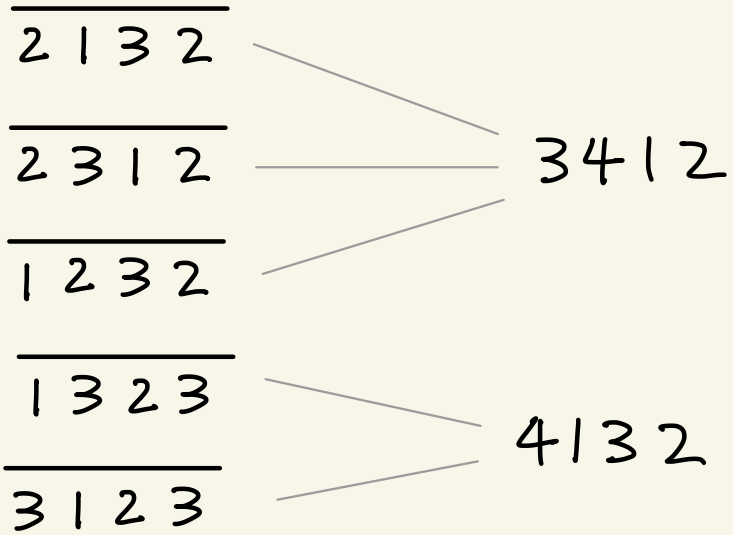
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Atoms

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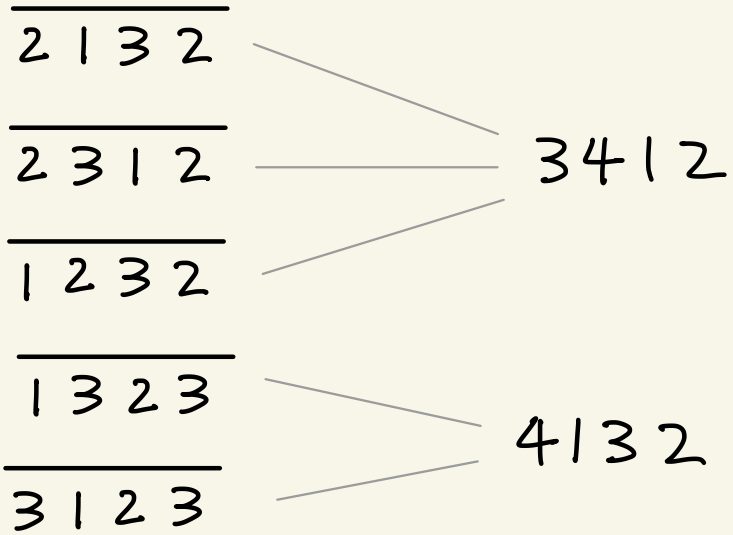
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$A_\mu(\pi)$ gives G_w expansion of G_π^μ.

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Goal: Give a combinatorial description of $A_\mu(\pi)$.

The set of <u>inverse atoms</u>, $A_\mu^{-1}(\pi) = \{w^{-1} : w \in A_\mu(\pi)\}$,

has nice a combinatorial description.

Involution Inverse Atoms

Thm [Can-Joyce-Wyser] [Hamaker-Marberg-Palowski]

Main idea: cycles dictate the relative orders of letters in any inverse atom.

Cycles	Order	Subword in inverse atom
(a,b)	$a > b$	ab
$(a,b)(c,d)$	$a > c$, $b > d$	$cdab$
	$a > c > d > b$	$cdab$ $cabd$ $abcd$
$(a,b)(c)$	$a > c > b$	cab abc

Ex. $\pi = 5327164$ $(5,1)(3,2)(7,4)(6)$

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$(5,1)(3,2) \Rightarrow 3251, 3512, 5132$

$(5,1)(7,4) \Rightarrow 5174$ $\quad (5,1)(6) \Rightarrow 516$

$(3,2)(7,4) \Rightarrow 3274$ $\quad (3,2)(6) \Rightarrow 326$

$(7,4)(6) \Rightarrow 674, 746$

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μ-Involution Inverse Atoms

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Ex. $\pi = 651|3|742$

Block	$B_1 : 651$	$B_2 : 3$	$B_3 : 742$
Cycles	$(6,1)(5)$	(3)	$(7,2)(4)$
$A^{-1}(B_i)$	$561, 615$	3	$472, 724$

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$$A_\mu^{-1}(\pi) = \left\{ 5613472,\ 5613724,\ 6153472,\ 6153724 \right\}$$

Something Interesting ...

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$6|53724$

Something Interesting ...

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$61537\underline{24} \in A_\mu^{-1}(\pi) \qquad \underline{26}47315 \in A_\mu(\pi)$

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$6153724 \in A_\mu^{-1}(\pi)$ $\quad 2647315 \in A_\mu(\pi)$

leading atom

Cycle notation gives leading inverse atom in reverse lexicographic order.

Strong Bruhat Order and Transition Equation

Strong Bruhat Order

$\pi \leq^{\mu} \tau$ orbit closure containment

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Thm [Timashev]

weak order $\Rightarrow$ Strong Bruhat Order

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Thm [Banerjee-Can-Joyce]

For $\pi, \tau \in I_\mu$, $\pi \leq^{\mu} \tau$ if and only if $\forall\, r \in R(\tau)$, $\exists\, r' \in R(\pi)$ s.t. r' is a subword of r and $\ell(\pi)+1 = \ell(\tau)$.

Ex.

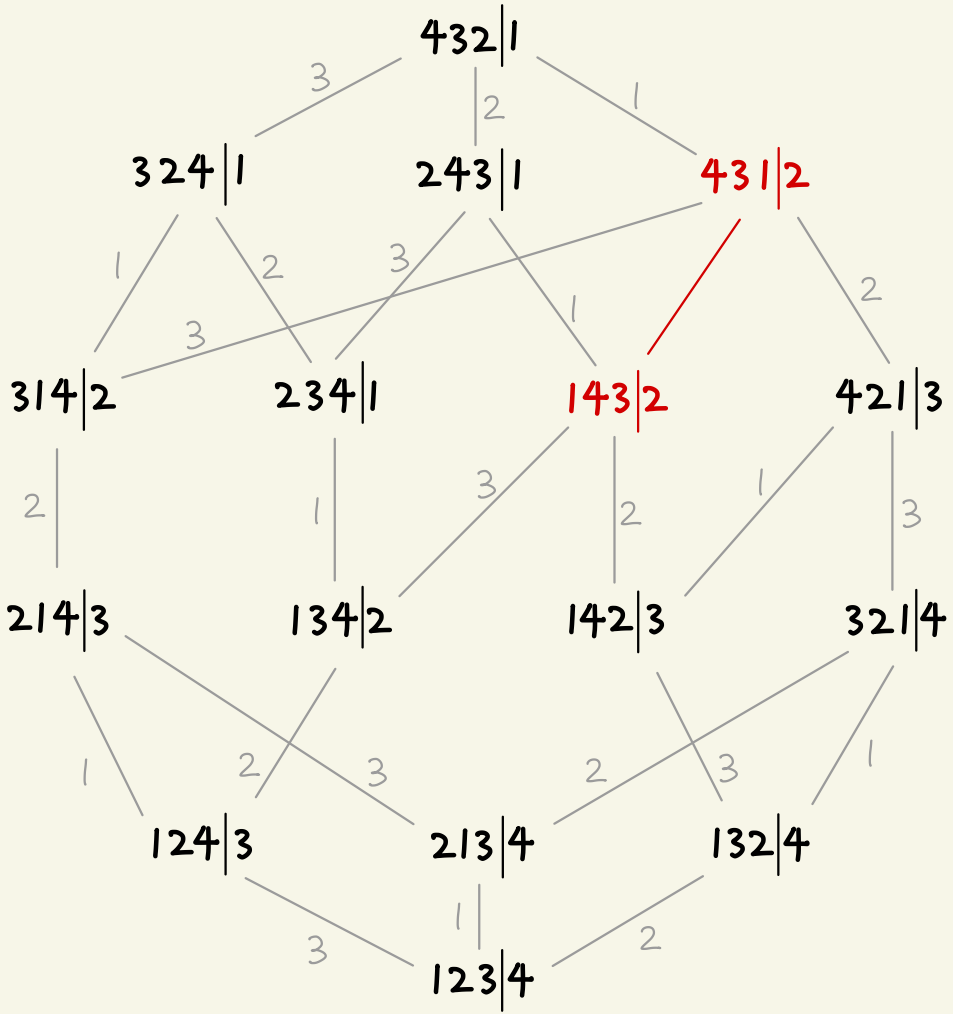
$\pi = 143|2 \quad \tau = 431|2$

$$R(\tau) = \left\{ \begin{matrix} 2132 & 2312 \\ 1232 & 1323 \\ 3123 \end{matrix} \right\}$$

$$R(\pi) = \{ 232 \quad 323 \}$$

$$\ell(\pi)+1 = \ell(\tau)$$

$$\Rightarrow \pi \lessdot^{\mu} \tau$$



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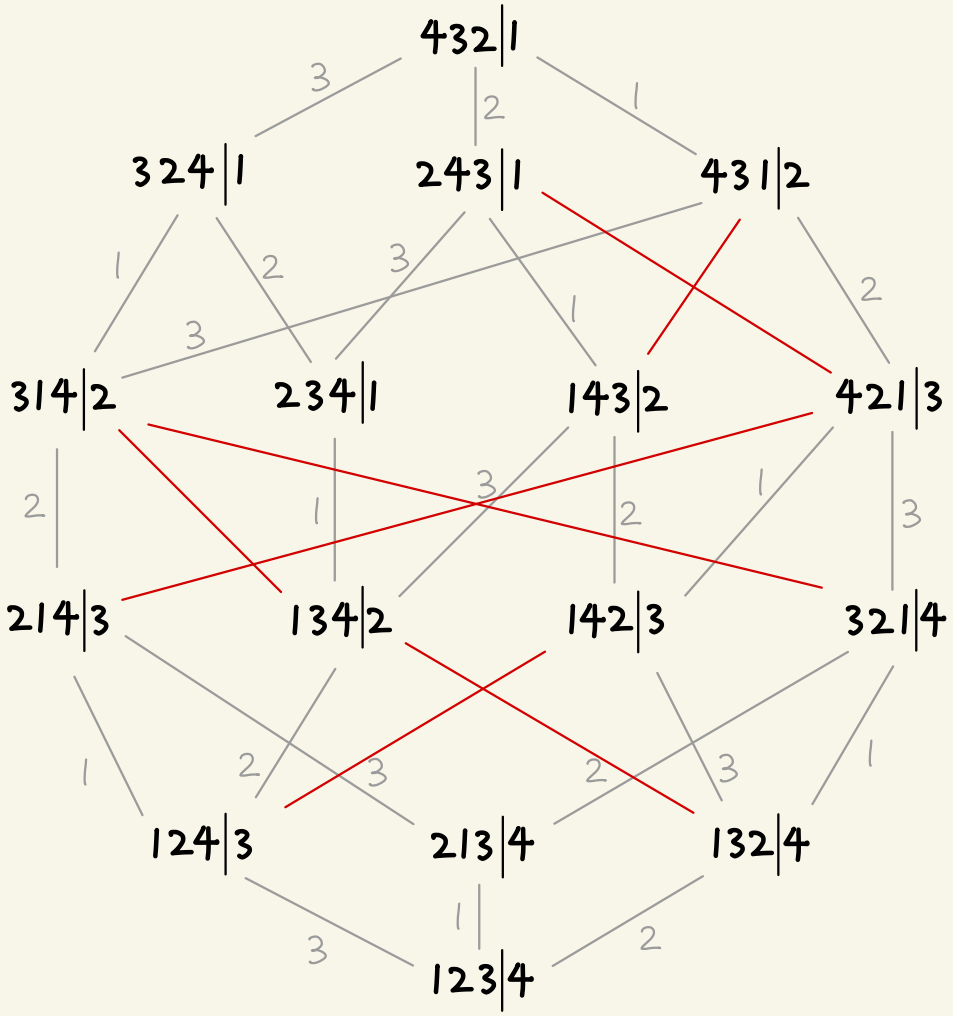
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Strong Bruhat Order

Def: Operator t_{ij}^{μ} on $\pi \in I_\mu$ $(i<j)$

- Swap values i and j in $Cyc(\pi)$
- Break $(a,b) \to (a)(b)$ if $a<b$
- Two special cases when i, j in the same block with $i<k<\ell<j$

$$(k,i)(j,\ell) \longrightarrow (j,i)(k)(\ell)$$

$$(\ell,i)(j,k) \longrightarrow (j,i)(\ell,k)$$

Ex.

$$t_{25}^{\mu}(2|176|543)$$

$$(2)|(1)(7,6)|(5,3)(4)$$

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$$= 5|176|234$$

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$t^{\mu}_{16}(5|4613|72)$

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$= t^{\mu}_{13}(5|4613|72)$

Strong Bruhat Order

Thm [C. - Hamaker] For $\pi, \tau \in I_\mu$,

$$\pi \lessdot^\mu \tau \quad \text{iff} \quad \begin{array}{l} t_{ij}^\mu(\pi) = \tau \quad \text{for some } i, j \\ \text{and} \quad \ell(\pi)+1 = \ell(\tau) \end{array}$$

Strong Bruhat Order

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Ex. $\pi = 2|176|543$, $\tau = 5|176|234$, $t_{25}^\mu(\pi) = \tau$

$\ell(\pi) = 10$, $\ell(\tau) = 11 \implies \pi \lessdot^\mu \tau$

Ex. $\pi = 5|4613|72$, $\tau = 5|6431|72$, $t_{16}^\mu(\pi) = \tau$

$\ell(\pi) = 11$, $\ell(\tau) = 12 \implies \pi \lessdot^\mu \tau$

Strong Bruhat Order

Thm [C.-Hamaker] For $\pi <^{\mu} \tau$,

$$A_{\mu}^{-1}(\tau) = \left\{ t_{ij} w : w \in A_{\mu}^{-1}(\pi),\ w <: t_{ij} w,\ t_{ij}^{\mu}(\pi) = \tau \right\}$$

Ex. $\pi = 2|176|543 \xrightarrow{t_{25}^{\mu}} 5|176|234 = \tau$

$A_{\mu}^{-1}(\pi) =$

$$2176453 \xrightarrow{t_{25}} \cancel{5176423}$$

$$2176534 \xrightarrow{t_{25}} 5176234 = A_{\mu}^{-1}(\tau)$$

Transition

Thm [C.-Hamaker] For $\pi \in I_\mu$ and $(j,i) \in Cyc(\pi)$,

$$2^{-\delta(i,j)} \cdot (X_i + X_j)\mathcal{G}_\pi^\mu = \sum_{\tau \in \Phi} \mathcal{G}_\tau^\mu - \sum_{\tau \in \Psi} \mathcal{G}_\tau^\mu$$

$$\Phi = \left\{ \tau = t_{ab}^\mu(\pi) : \pi \lessdot^\mu \tau \,,\ a \in \{i,j\} \right\}$$

$$\Psi = \left\{ \tau = t_{ab}^\mu(\pi) : \pi \lessdot^\mu \tau \,,\ b \in \{i,j\} \right\}$$

Transition $\pi = 34|521|6$ $(3)(4)|(5,1)(2)|(6)$

Bruhat covers: t_{16}^{μ} t_{26}^{μ} t_{34}^{μ} t_{45}^{μ} t_{56}^{μ}

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Transition $\pi = 34|521|6$ $\quad (3)(4)|(5,1)(2)|(6)$

Bruhat covers: t_{16}^{μ} t_{26}^{μ} t_{34}^{μ} t_{45}^{μ} t_{56}^{μ}

$$X_4 \cdot \mathcal{G}_{\pi}^{\mu} = \mathcal{G}_{t_{45}^{\mu}(\pi)}^{\mu}$$

Transition $\pi = 34|521|6$ $(3)(4)|(5,1)(2)|(6)$

Bruhat covers: t^u_{16} t^u_{26} t^u_{34} t^u_{45} t^u_{56}

$$X_4 \cdot \mathcal{G}^u_\pi = \mathcal{G}^u_{t^u_{45}(\pi)} - \mathcal{G}^u_{t^u_{34}(\pi)}$$

Transition $\pi = 34|521|6$ $\quad (3)(4)|(5,1)(2)|(6)$

Bruhat covers: t_{16}^{μ} t_{26}^{μ} t_{34}^{μ} t_{45}^{μ} t_{56}^{μ}

$$X_4 \cdot \mathcal{G}_{\pi}^{\mu} = \mathcal{G}_{t_{45}^{\mu}(\pi)}^{\mu} - \mathcal{G}_{t_{34}^{\mu}(\pi)}^{\mu}$$

$$(X_1 + X_5) \cdot \mathcal{G}_{\pi}^{\mu} =$$

Transition $\pi = 34|521|6$ $\quad (3)(4)|(5,1)(2)|(6)$

Bruhat covers: t^{μ}_{16} t^{μ}_{26} t^{μ}_{34} t^{μ}_{45} t^{μ}_{56}

$$X_4 \cdot \mathcal{G}^{\mu}_{\pi} = \mathcal{G}^{\mu}_{t^{\mu}_{45}(\pi)} - \mathcal{G}^{\mu}_{t^{\mu}_{34}(\pi)}$$

$$(X_1 + X_5) \cdot \mathcal{G}^{\mu}_{\pi} = \mathcal{G}^{\mu}_{t^{\mu}_{16}(\pi)} + \mathcal{G}^{\mu}_{t^{\mu}_{56}(\pi)}$$

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RMK: $X_1 \cdot \mathcal{G}_{\pi}^{\mu} \notin \mathbb{Z}[\mathcal{G}_{\tau}^{\mu}]$

Thm For $\pi \in I_\mu$ with blocks $B_1 \cdots B_k$,

$$A_\mu^{-1}(\pi) = \left\{ Q_1 Q_2 \cdots Q_k : Q_i \in A^{-1}(B_i) \right\}$$

Thm For $\pi, \tau \in I_\mu$, $\pi \lessdot^\mu \tau$ iff

$t_{ij}^\mu(\pi) = \tau$ for some i, j

and $\ell(\pi)+1 = \ell(\tau)$

Thm For $\pi \lessdot^\mu \tau$,

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Thank You !!