Constructing Maximal Pipedreams of Double Grothendieck Polynomials

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Overview

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4. Maximal pipedreams of double Grothendieck polynomials
5. Algorithm and main theorems

Schubert Calculus

Hilbert's 15th problem: Rigorous foundation of Hermann Schubert's enumerations

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Schubert polynomials $\mathfrak{S}_w(\mathbf{x})$

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Double Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$

Permutations

$[n] = \{1, 2, \ldots, n\}$

One-line notation:

$$w = 521463 \in S_6$$

$$\begin{array}{lll} 1 \to 5 & 2 \to 2 & 3 \to 1 \\ 4 \to 4 & 5 \to 6 & 6 \to 3 \end{array}$$

long element $w_0 = n \quad n-1 \cdots 2 \quad 1.$

Permutations

Inversions of permutations:

$w = 521463$

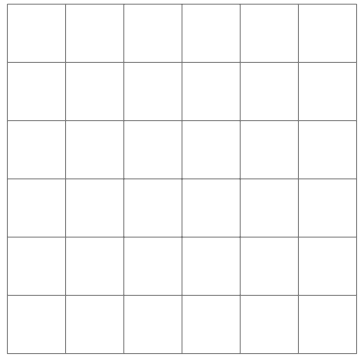
$$\mathrm{Inv}(w) = \{(1,2),(1,3),(1,4),(1,6),(2,3),(4,6),(5,6)\}$$

Length of $w = \ell(w) = |\mathrm{Inv}(w)| = 7$.

invcode(w) = **Lehmer code**$(w) = (4,1,0,1,1,0)$.

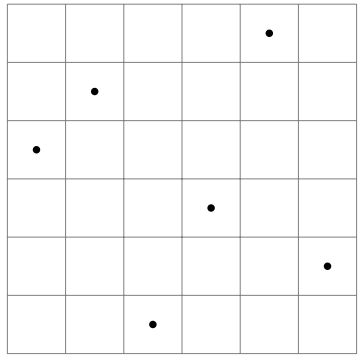
Permutations

Rothe Diagram of $w = 521463$:



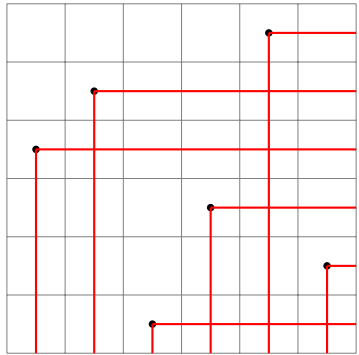
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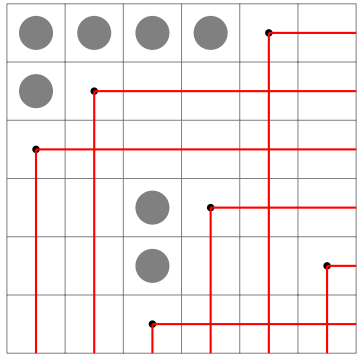
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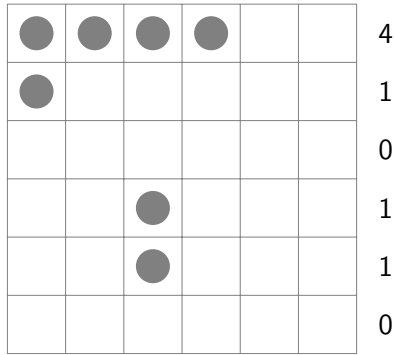
Permutations

Rothe Diagram of $w = 521463$:



Permutations

Rothe Diagram of $w = 521463$, $\text{invcode}(w) = (4, 1, 0, 1, 1, 0)$:



Grothendieck Polynomials and Pipedreams

Grothendieck polynomials

$f \in \mathbb{Z}[x_1, \ldots, x_n]$
Divided Difference Operators:

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}}$$
$$\overline{\partial_i}(f) = \partial_i(f - x_{i+1} f)$$

s_i switches the variable x_i and x_{i+1}.

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$$\partial_1(x_1^2) = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$$
$$\overline{\partial_1}(x_1^2) = \partial_1(x_1^2 - x_1^2 x_2) = \frac{(x_1^2 - x_1^2 x_2) - (x_2^2 - x_1 x_2^2)}{x_1 - x_2} = x_1 + x_2 - x_1 x_2$$

Grothendieck polynomials

Grothendieck polynomials:

- Lascoux and Schützenberger
- Classes of the structure sheaves of Schubert varieties
- Labeled by permutations
- Defined recursively

Grothendieck polynomials

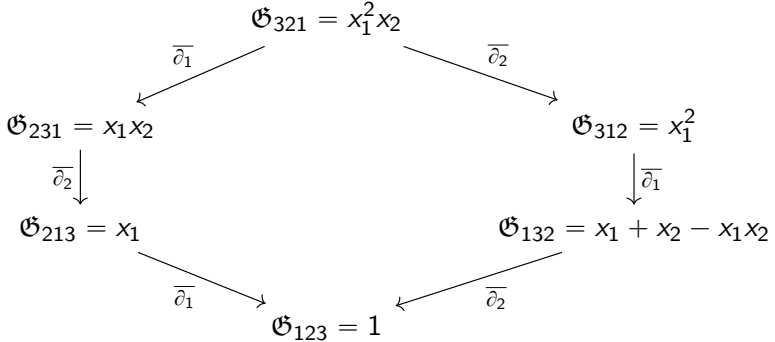
Grothendieck polynomials:

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$$\mathfrak{G}_w(\mathbf{x}) := \begin{cases} x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2 x_{n-1}^1 & w = w_0 \\ \overline{\partial_i}(\mathfrak{G}_{ws_i}(\mathbf{x})) & w(i) < w(i+1) \end{cases}$$

where s_i switches the i and $i+1$ position of w.

Grothendieck polynomials



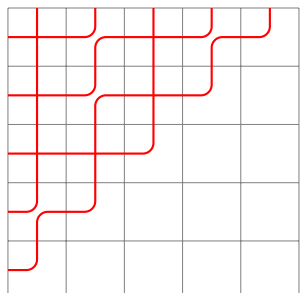
Pipedreams

Are there any other ways to compute Grothendieck polynomials?

Pipedreams

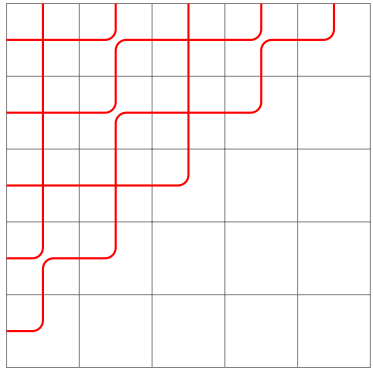
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Pipedream = tiling with , , and .

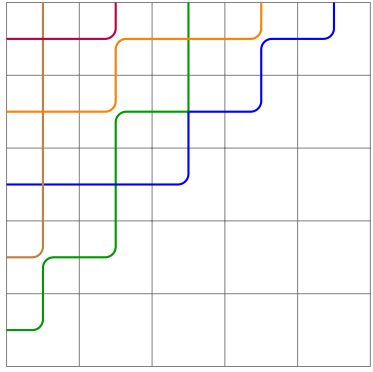


wt(P) = number of  in each row = $(2, 2, 2, 0, 0)$

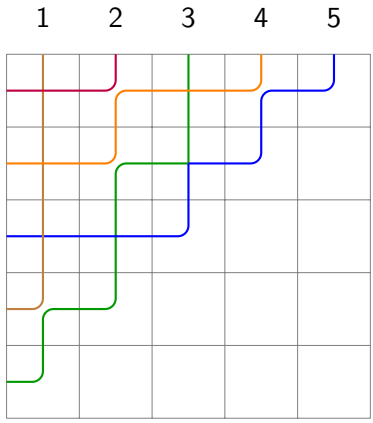
Pipedreams



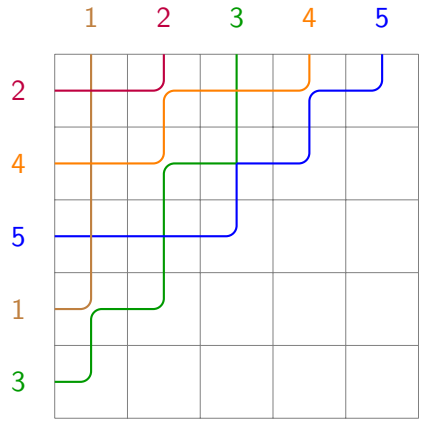
Pipedreams



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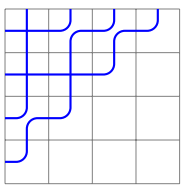
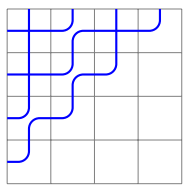
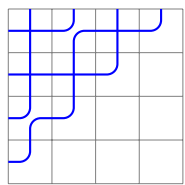
Pipedreams



Pipedreams

$\mathrm{PD}(w)$ = set of all pipedreams associated with w

$w = 2413$, $\mathrm{PD}(w) =$

Pipedreams

Theorem (Knutson-Miller)

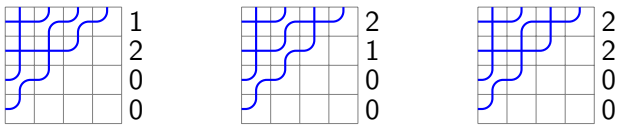
$$\mathfrak{G}_w(\mathbf{x}) = \sum_{P \in \mathrm{PD}(w)} (-1)^{|\mathrm{wt}(P)|-\ell(w)} \mathbf{x}^{\mathrm{wt}(P)}$$

Pipedreams

Theorem (Knutson-Miller)

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$\mathrm{PD}(2413) =$



$$\implies \mathfrak{G}_{2413} = x_1x_2^2 + x_1^2x_2 - x_1^2x_2^2$$

Pipedreams

Question:
How do we generate all the pipedreams of w?
How do we find one pipedream of w?

Pipedreams

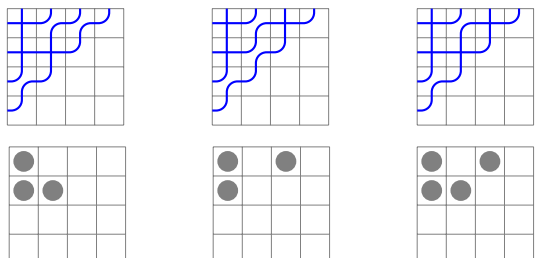
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Answer:
The left-justified Rothe diagram of w is in $\mathrm{PD}(w)$.
All pipedreams in $\mathrm{PD}(w)$ can be achieved from $\overleftarrow{\mathrm{Rothe}(w)}$ through ladder moves and K-ladder moves.

Pipedreams

Convention:  = ●.

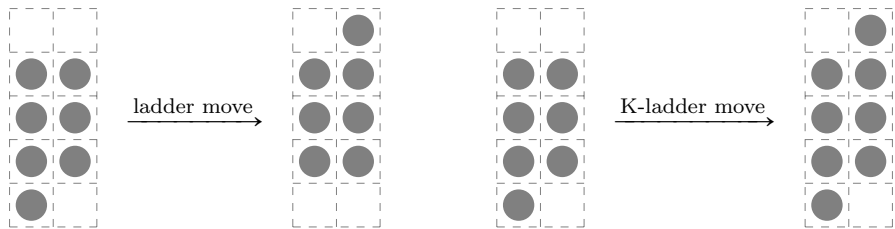
PD(2413) =



wt(P) = number of ● in each row

Pipedreams

(K-)ladder moves:

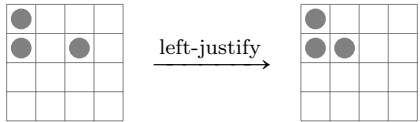


Pipedreams

Find all the pipedreams for $w = 2413$.

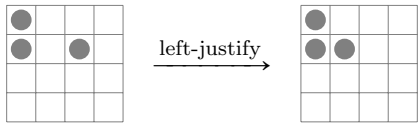
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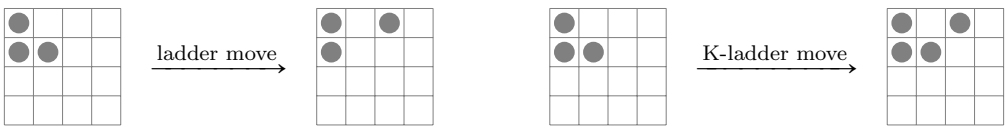
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Rothe(2413) and $\overleftarrow{\text{Rothe}(2413)}$:



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Maximal Pipedreams of Double Grothendieck Polynomials

Maximal pipedreams

Double Grothendieck polynomials:

- Generalizations of Grothendieck polynomials in $2n$ variables
- Schubert classes in the torus-equivariant K-theory of flag varieties

Maximal pipedreams

Double Grothendieck polynomials:

Generalizations of Grothendieck polynomials in $2n$ variables

Schubert classes in the torus-equivariant K-theory of flag varieties

$$\mathfrak{G}_w(\mathbf{x},\mathbf{y}) := \begin{cases} \prod_{i+j\leqslant n}(x_i + y_j - x_i y_j) & w = w_0 \\ \overline{\partial}_i(\mathfrak{S}_{ws_i}(\mathbf{x},\mathbf{y})) & w(i) < w(j+1) \end{cases}$$

Theorem (Knutson-Miller)

$$\mathfrak{G}_w(\mathbf{x},\mathbf{y}) = \sum_{P\in \mathrm{PD}(w)} (-1)^{|\mathrm{wt}(P)|-\ell(w)} \prod_{(i,j)\in P} (x_i + y_j - x_i y_j)$$

$\mathfrak{G}_w(\mathbf{x},0) = \mathfrak{G}_w(\mathbf{x})$.

Maximal pipedreams

Definition (Pechenik-Speyer-Weigandt)

$$\text{rajcode}(w) = \text{degree of leading monomial of } \mathfrak{G}_w(\mathbf{x})$$

with lexicographical order where $x_n > \cdots > x_1$.

Maximal pipedreams

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Theorem (Pechenik-Speyer-Weigandt)

$$\textit{Leading monomial of } \mathfrak{G}_w(\mathbf{x}, \mathbf{y}) = \mathbf{x}^{\text{rajcode}(w)}\mathbf{y}^{\text{rajcode}(w^{-1})}$$
$$\textit{Coefficient} = \pm 1$$

with lexicographical order where $x_n > \cdots > x_1$, *and* $y_n > \cdots > y_1$.

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$$x_1x_2^3x_3^2x_5x_6 > x_1x_2^3x_3^2x_4x_6 > x_1^2x_2^3x_3x_4x_6$$

Maximal pipedreams

There exists a unique pipedream in $\mathrm{PD}(w)$ whose row weight is $\mathrm{rajcode}(w)$ and column weight is $\mathrm{rajcode}(w^{-1})$. We call this the **maximal pipedream** of $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$.

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Remark 7.2 (Pechenik Speyer Weigandt):

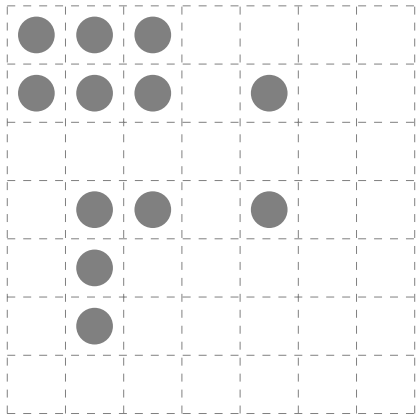
We find it frustrating that we do not have a direct recipe for the maximal pipe dream in terms of w.

Maximal pipedreams

Find rajcode(4617352) and rajcode(4617352^{-1}).

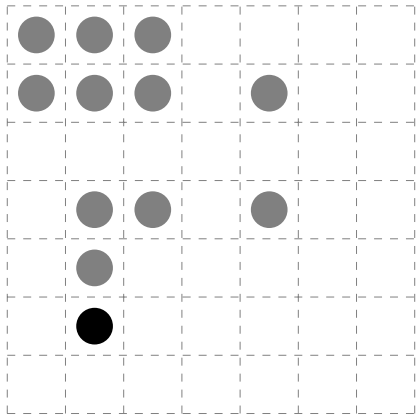
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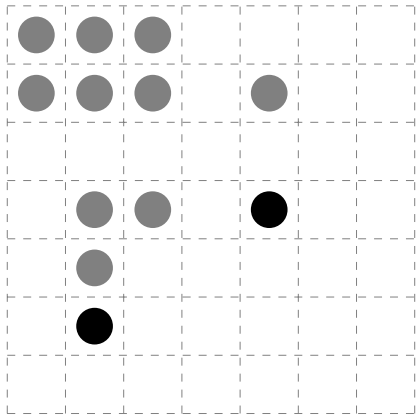
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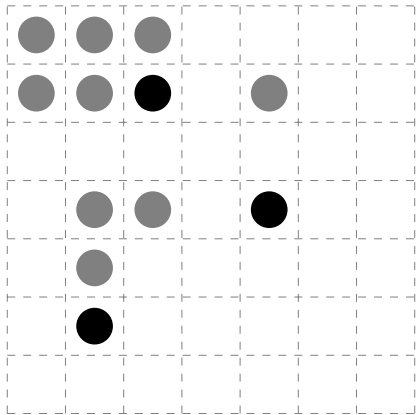
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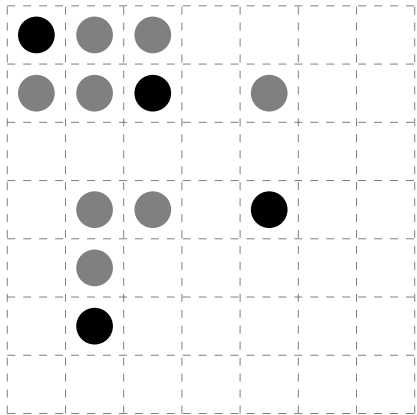
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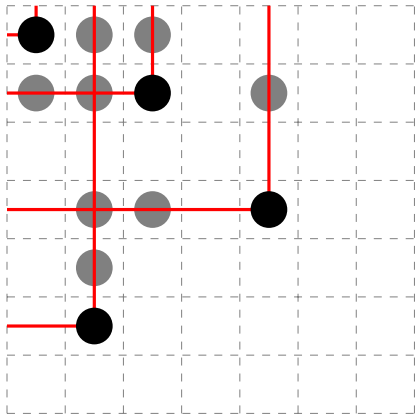
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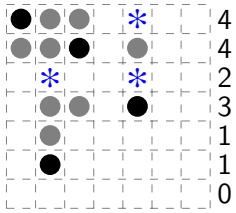
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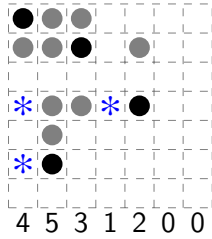
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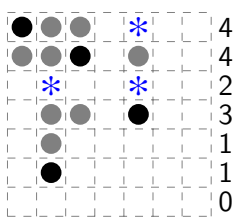
Define $\mathrm{Snow}(w)$ and $\overleftarrow{\mathrm{Snow}(w)}$:

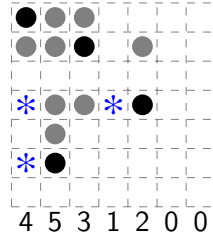




Maximal pipedreams

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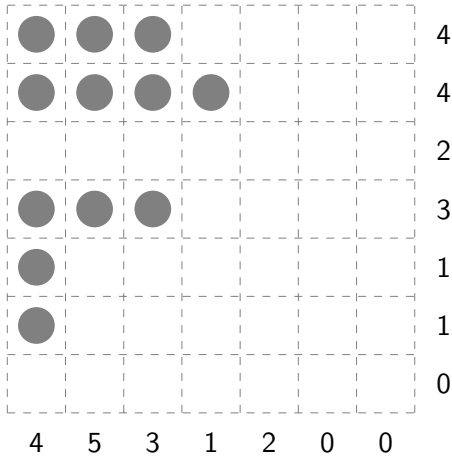




Proposition (Pan-Yu)

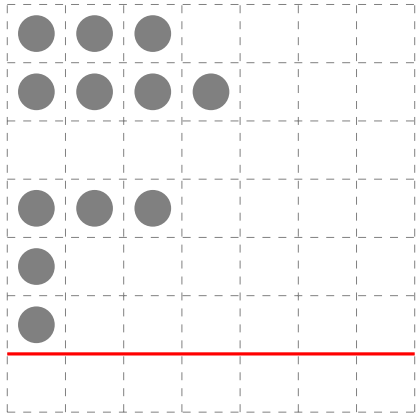
The row weight of $\mathrm{Snow}(w)$ *is* $\mathrm{rajcode}(w)$. *The column weight of* $\overleftarrow{\mathrm{Snow}(w)}$ *is* $\mathrm{rajcode}(w^{-1})$.

Maximal pipedreams

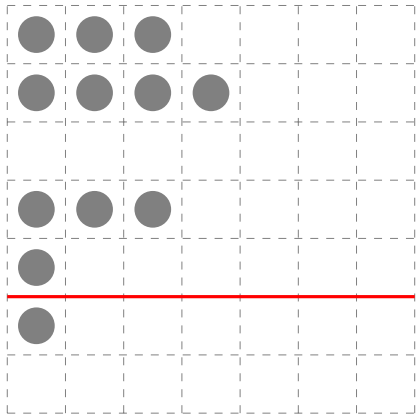


Algorithm and Main Theorems

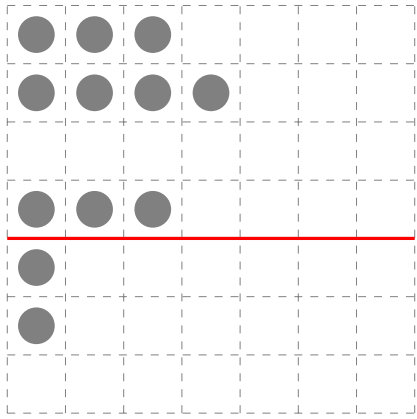
Algorithm



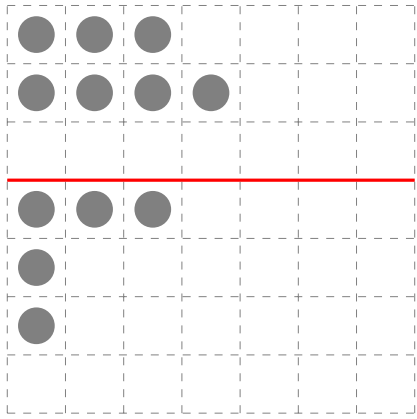
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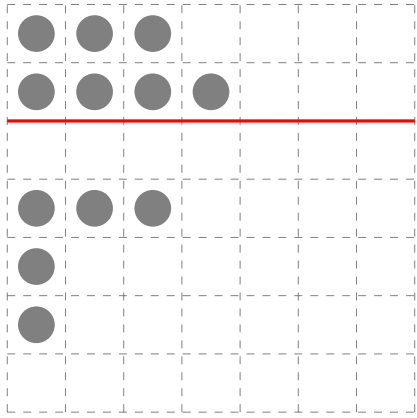
Algorithm



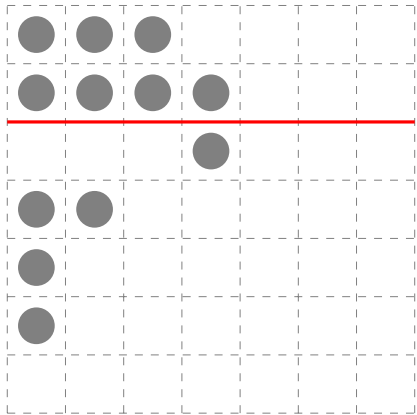
Algorithm



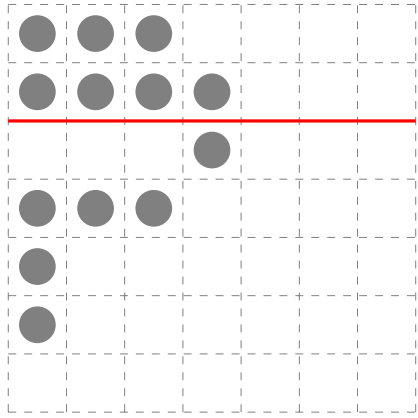
Algorithm



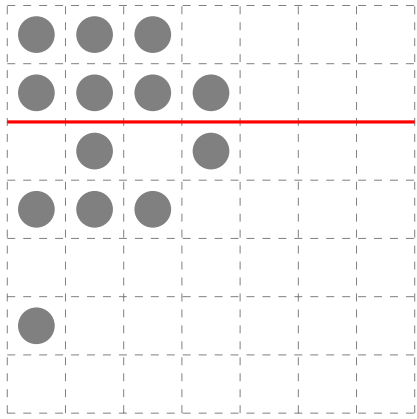
Algorithm



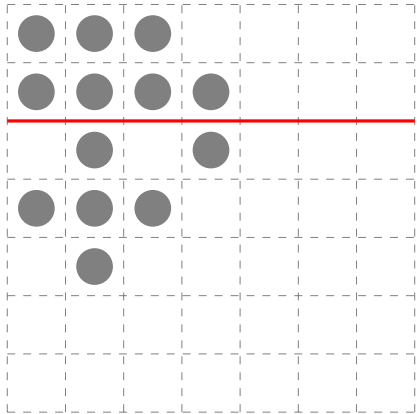
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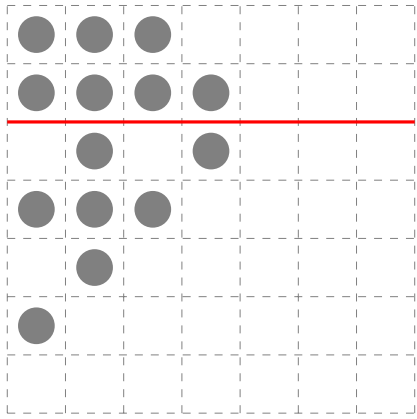
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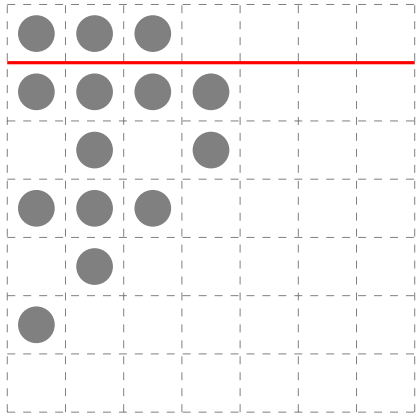
Algorithm



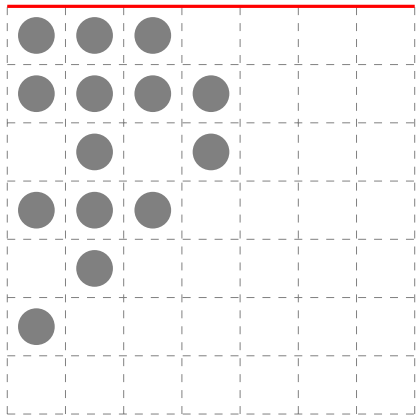
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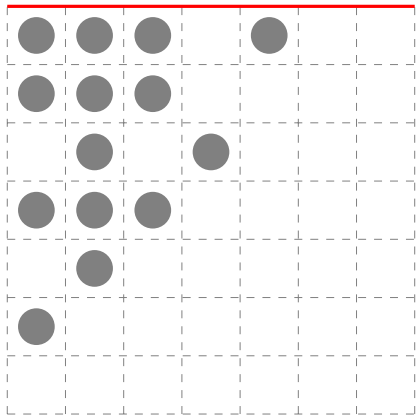
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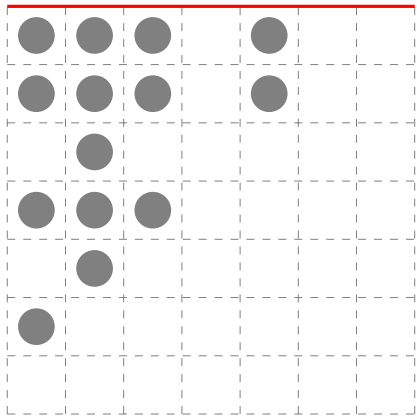
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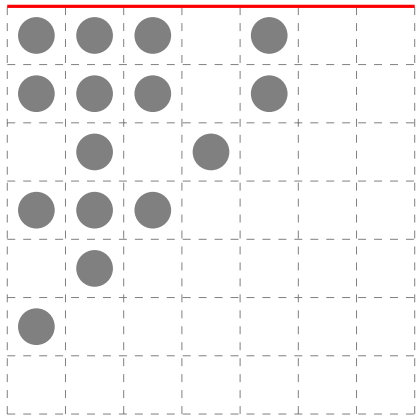
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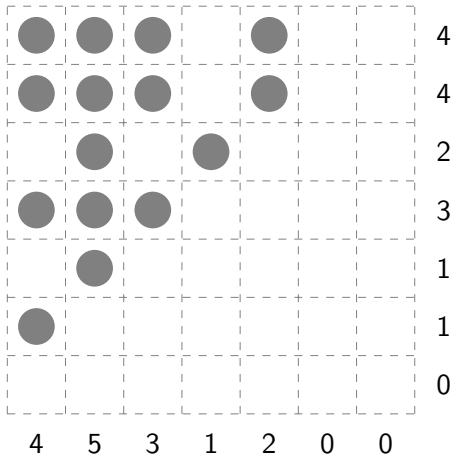
Algorithm



Algorithm



Algorithm



Main Theorems

$\widehat{P}(w)$ = result of our algorithm

Theorem (C.-Yu)

$\widehat{P}(w)$ has row weight rajcode(w) *and column weight* rajcode(w^{-1}). *Equivalently, $\widehat{P}(w)$ is the unique maximal pipedream of $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$.*

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Theorem (C.-Yu)

During our algorithm, the pipedream right after the ith K-ladder move has row weight equal to the degree of the leading monomial of the $i + \ell(w)$ degree homogeneous components of $\mathfrak{G}_w(\mathbf{x})$.

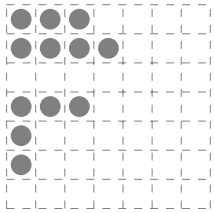
Main Theorems

Proposition (Dreyer-Mészáros-St. Dizier)

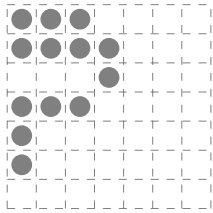
$m_0, \ldots, m_r =$ *leading monomials of* $\mathfrak{G}_w(\mathbf{x})$

$$\begin{aligned}
\mathbf{x}^{\mathsf{invcode}(w)} = m_0 &= x_1^3 x_2^4 x_3^0 x_4^3 x_5 x_6 \\
m_1 &= x_1^3 x_2^4 x_3 x_4^3 x_5 x_6 \\
m_2 &= x_1^3 x_2^4 x_3^2 x_4^3 x_5 x_6 \\
\mathbf{x}^{\mathsf{rajcode}(w)} = m_3 &= x_1^4 x_2^4 x_3^2 x_4^3 x_5 x_6
\end{aligned}$$

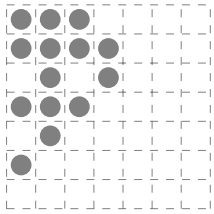
Main Theorems



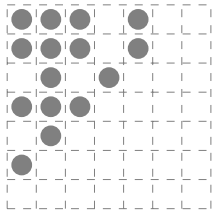
$m_0 = x_1^3 x_2^4 x_3^0 x_4^3 x_5 x_6$



$m_1 = x_1^3 x_2^4 x_3 x_4^3 x_5 x_6$



$m_2 = x_1^3 x_2^4 x_3^2 x_4^3 x_5 x_6$



$m_3 = x_1^4 x_2^4 x_3^2 x_4^3 x_5 x_6$

Main Theorems

Permutations in S_n are in bijection with weak compositions under a staircase through taking their invcodes.

Main Theorems

Permutations in S_n are in bijection with weak compositions under a staircase through taking their invcodes.

$$
\begin{aligned}
4617352 &\longrightarrow (3,4,0,3,1,1,0)\\
514362 &\longleftarrow (4,0,3,1,1,0)\\
15342 &\longleftarrow (0,3,1,1,0)\\
4231 &\longleftarrow (3,1,1,0)\\
231 &\longleftarrow (1,1,0)\\
21 &\longleftarrow (1,0)\\
1 &\longleftarrow (0)
\end{aligned}
$$

Main Theorems

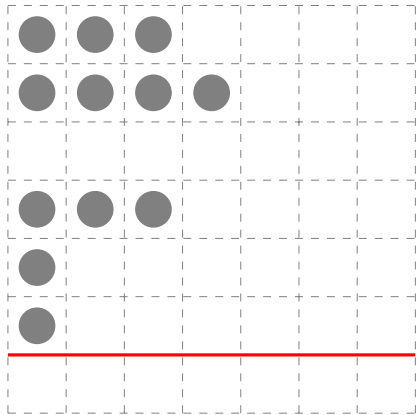
Permutations in S_n are in bijection with weak compositions under a staircase through taking their invcodes.

$$\begin{aligned} 4617352 &\longrightarrow (3,4,0,3,1,1,0) \\ 514362 &\longleftarrow (4,0,3,1,1,0) \\ 15342 &\longleftarrow (0,3,1,1,0) \\ 4231 &\longleftarrow (3,1,1,0) \\ 231 &\longleftarrow (1,1,0) \\ 21 &\longleftarrow (1,0) \\ 1 &\longleftarrow (0) \end{aligned}$$

Corollary (C.-Yu)

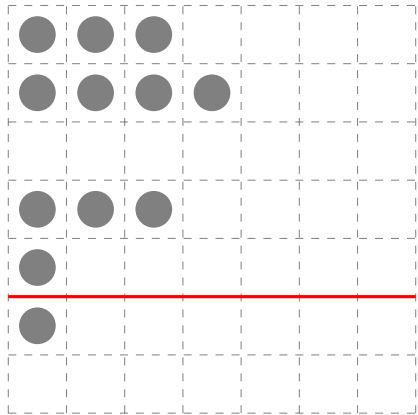
After each iteration in our algorithm, the diagrams under the bar are maximal pipedreams of the permutations in this sequence.

Main Theorems



1

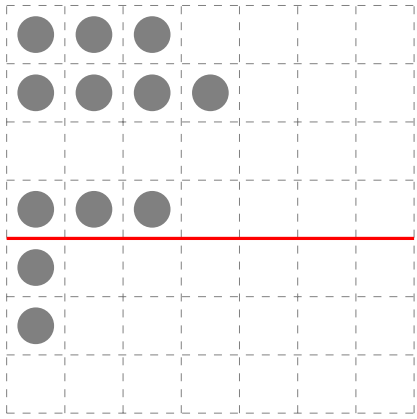
Main Theorems



21

1

Main Theorems

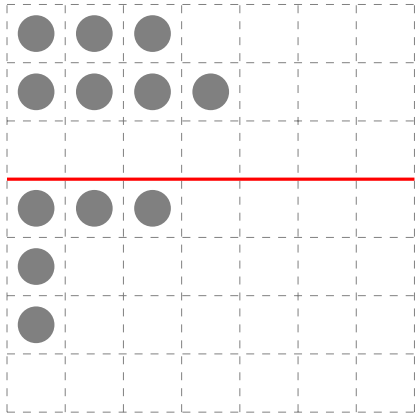


231

21

1

Main Theorems



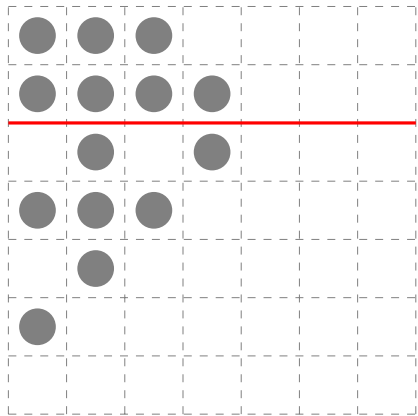
4231

231

21

1

Main Theorems



15342

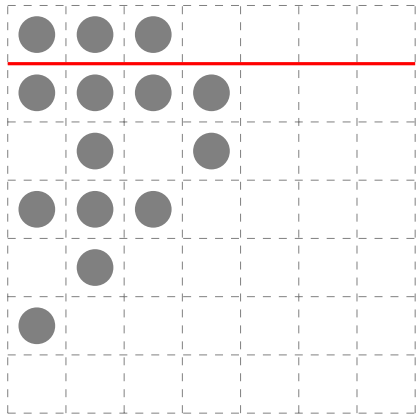
4231

231

21

1

Main Theorems



514362

15342

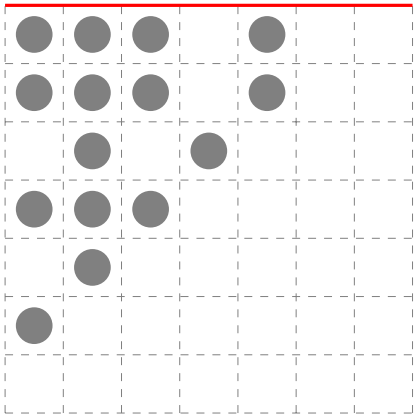
4231

231

21

1

Main Theorems



4617352

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1

Thank You!

Constructing maximal pipedreams of double Grothendieck polynomials
arXiv:2312.01250, 2023, with Tianyi Yu