Grothendieck Polynomials of Inverse Fireworks Permutations

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GSCC @ USC

Outline

1. Inverse Fireworks Permutations
2. Grothendieck Polynomials (Pipe Dreams)
3. Bumpless Vertical-less Pipe Dreams
4. Going-Up for Inverse Fireworks

Inverse Fireworks

Fireworks: The decreasing runs are increasing.

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3 1 6 7 5 4 2

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Inverse Fireworks

Why are they called

f*reworks?

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Inverse Fireworks

Fireworks ① 3__12 avoiding

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② Bell number $B(n)$

$3167542 \longleftrightarrow 1,3|2,7,4,5|6$

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Fireworks $\xleftrightarrow{\text{Inverse}}$ Inverse Fireworks

Grothendieck Polynomials and Pipe Dreams

Grothendieck Polynomials

K-theoretic analogues of Schubert polynomials

$$G_w(x) = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = w_0 \\ \overline{\partial_i}\left(G_{ws_i}(x)\right) & \text{if } w(i) < w(i+1) \end{cases}$$

Grothendieck Polynomials

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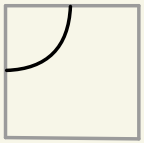
Combinatorial formulas:

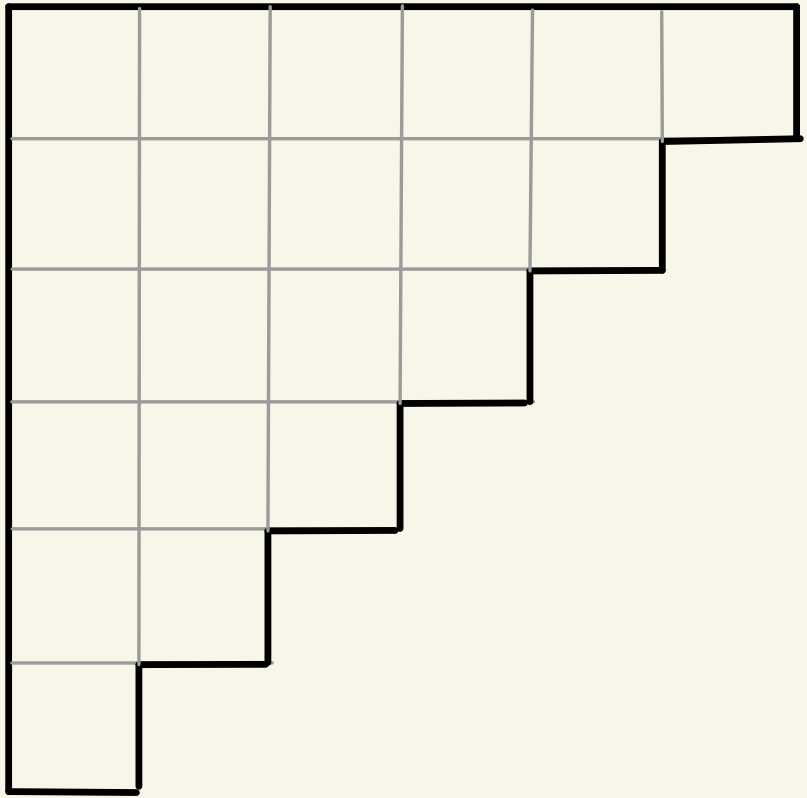
Pipe Dreams and Bumpless Pipe Dreams

Pipe Dreams





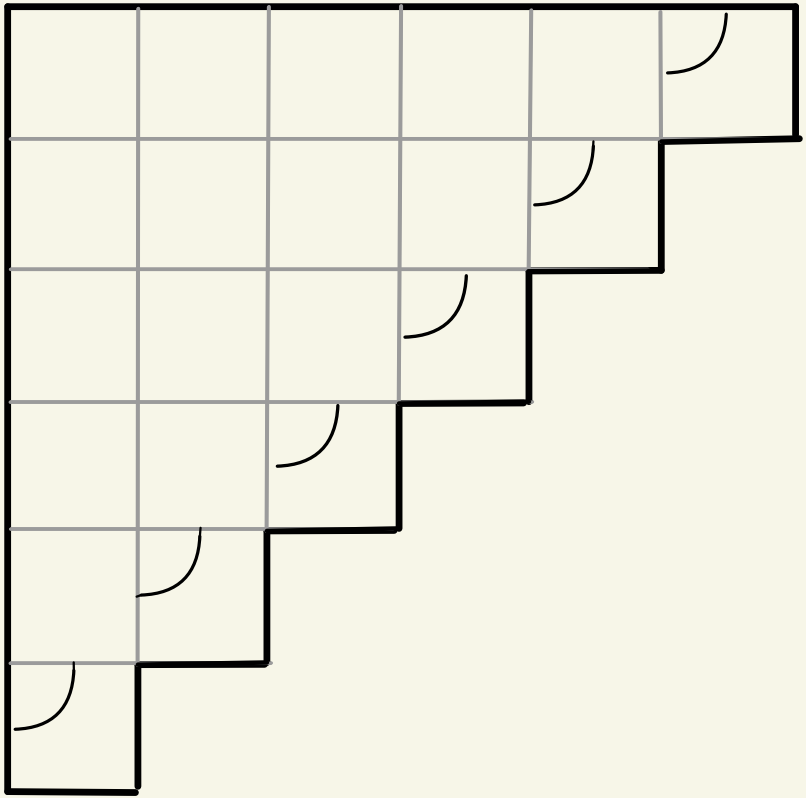




Pipe Dreams

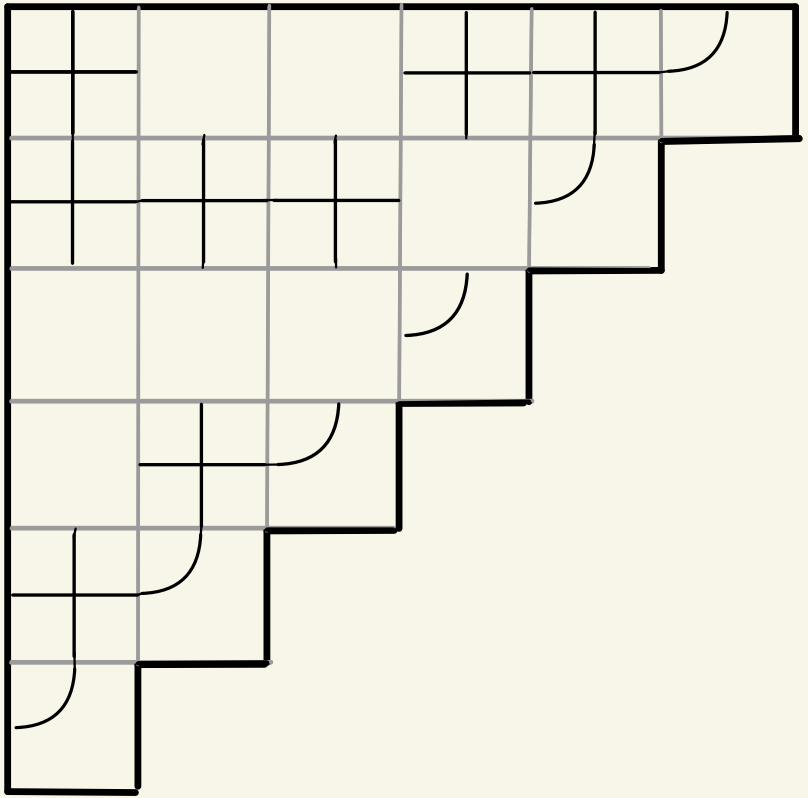




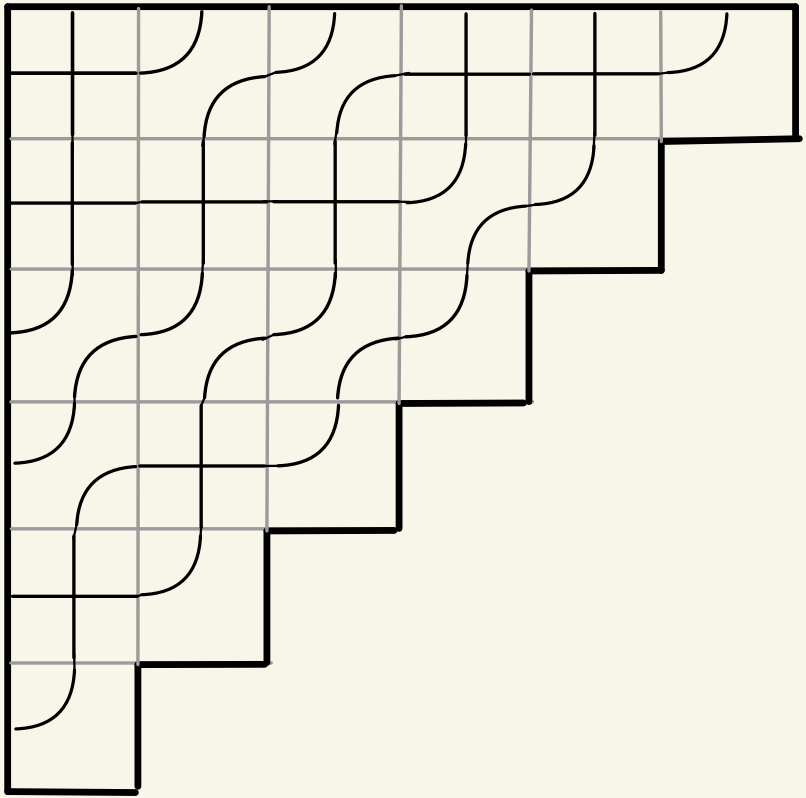


Pipe Dreams

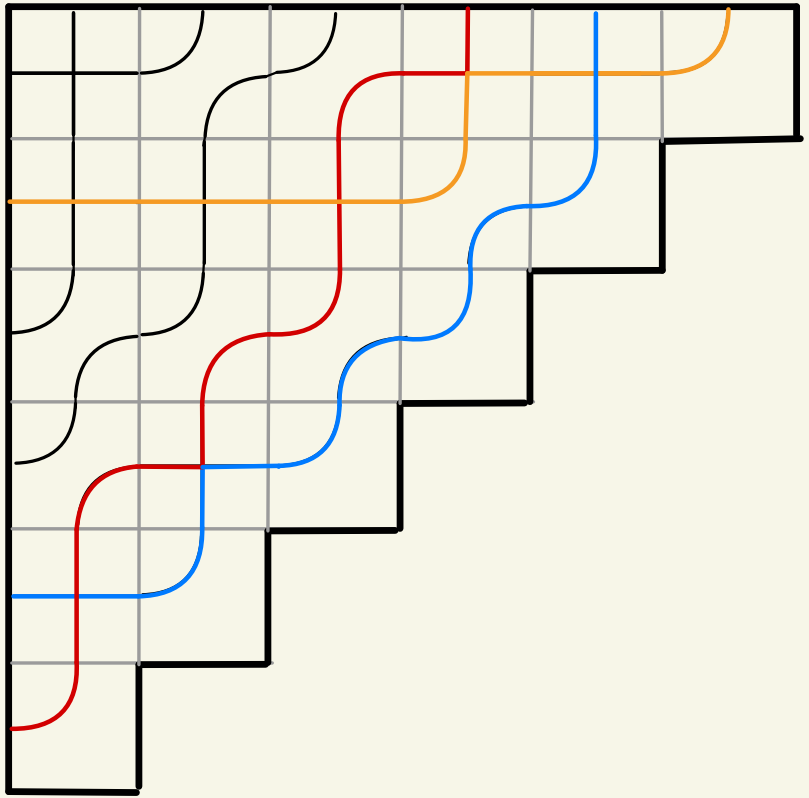




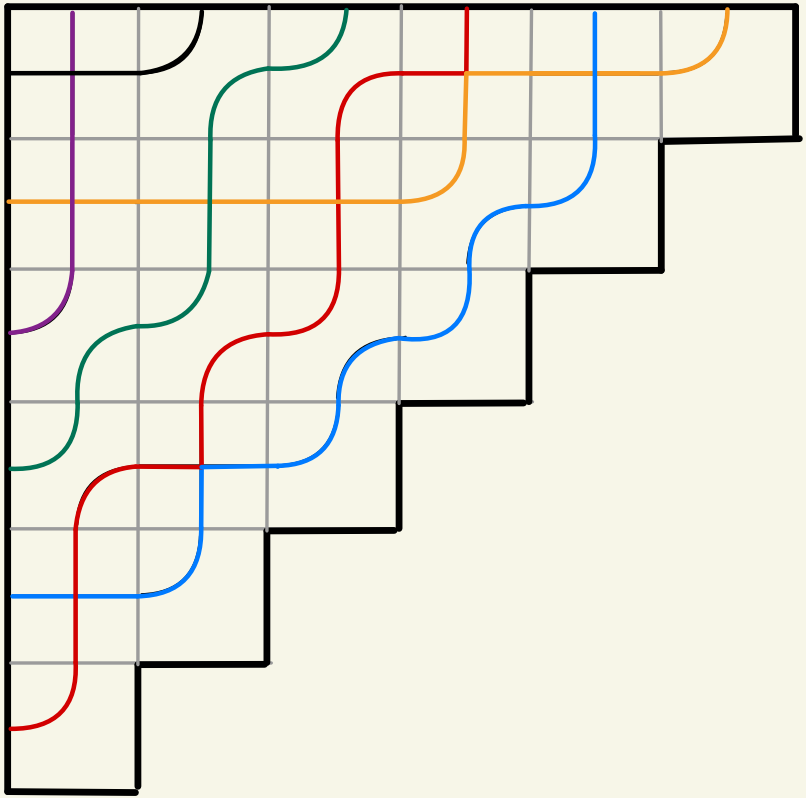
Pipe Dreams



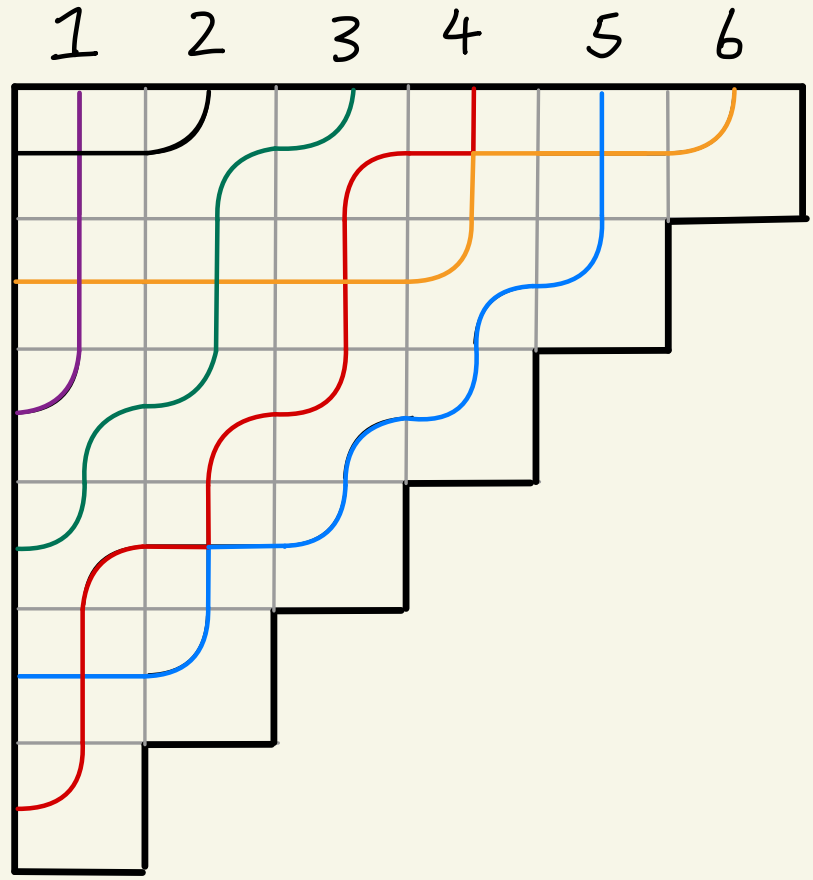
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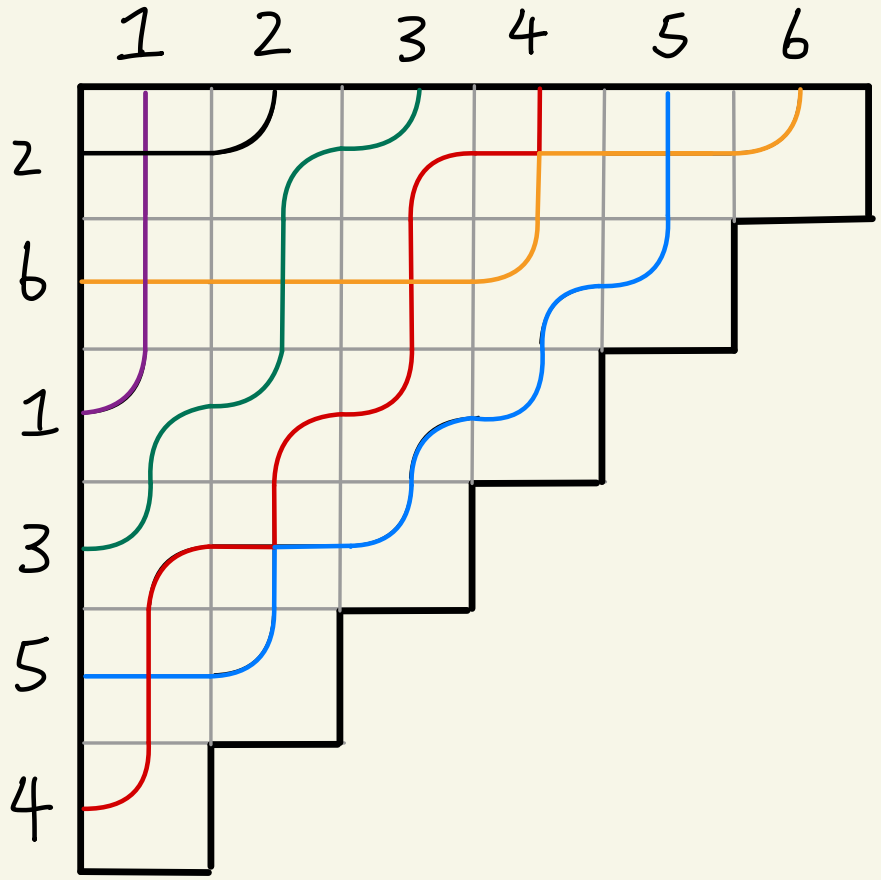
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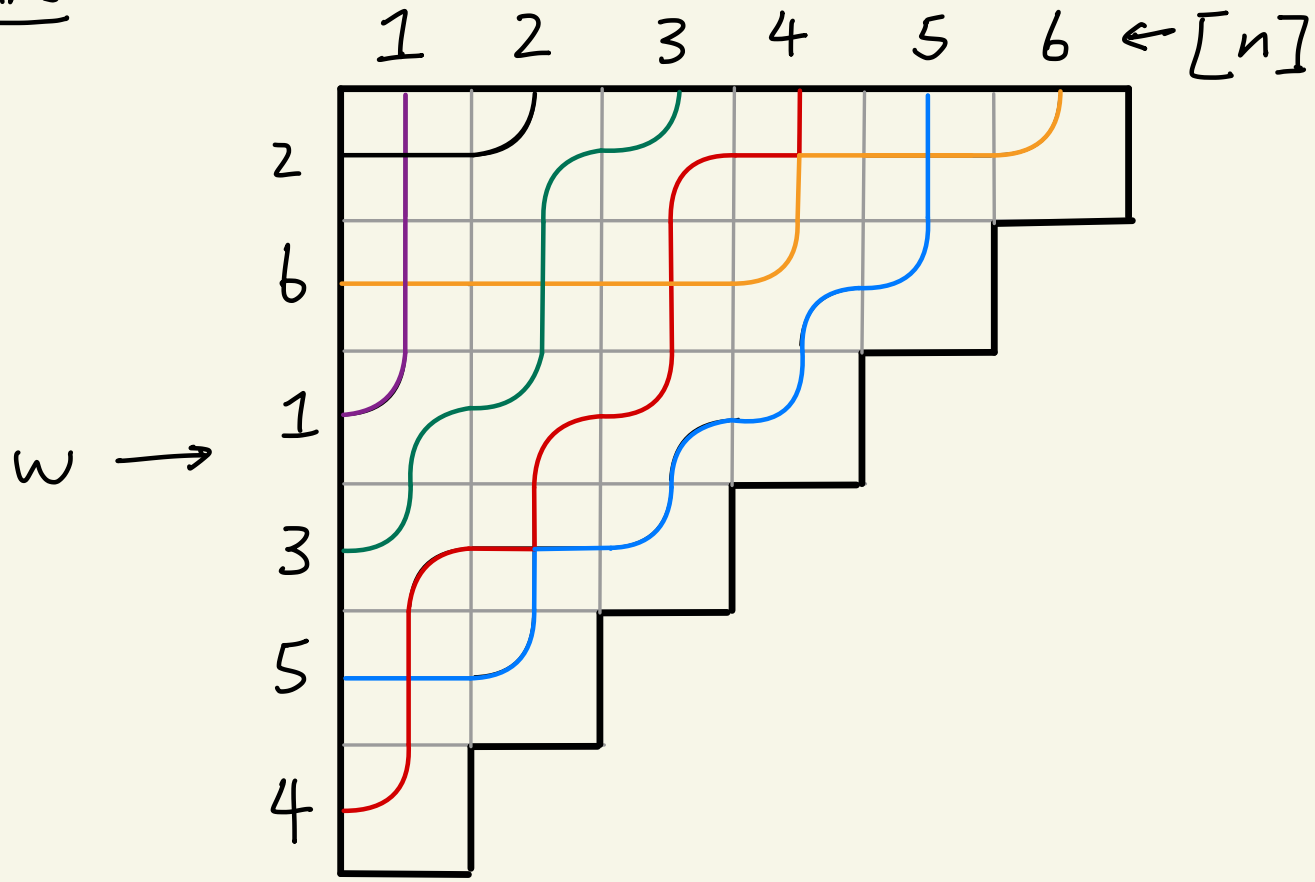
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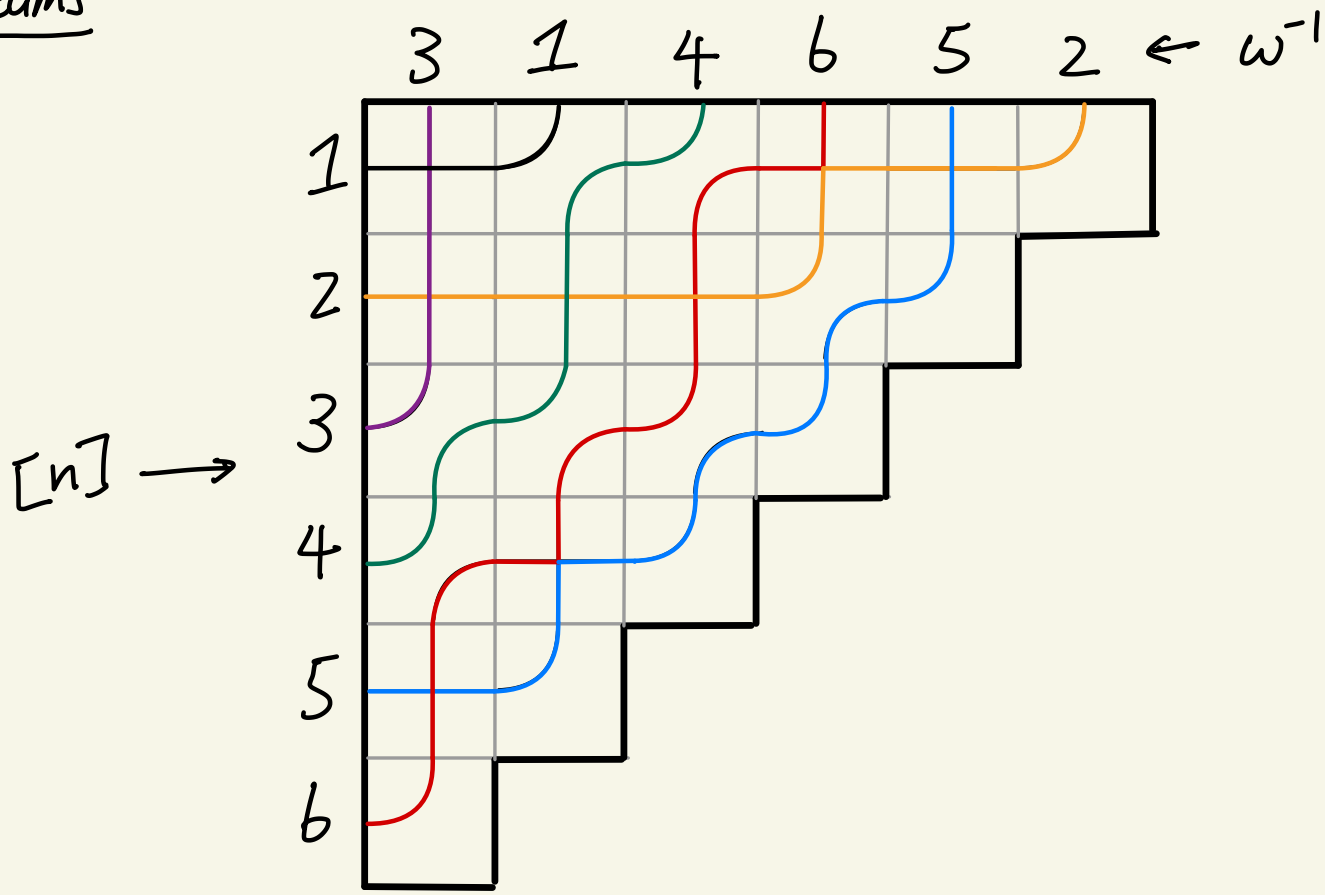
Pipe Dreams



Pipe Dreams

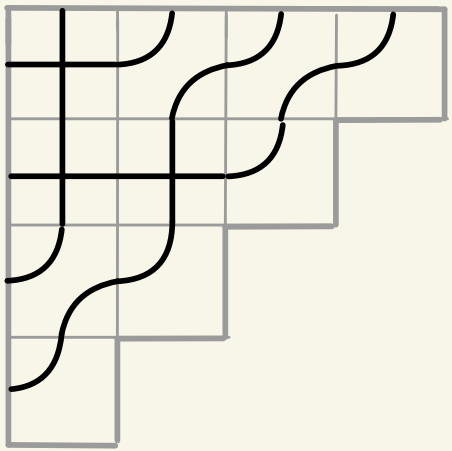


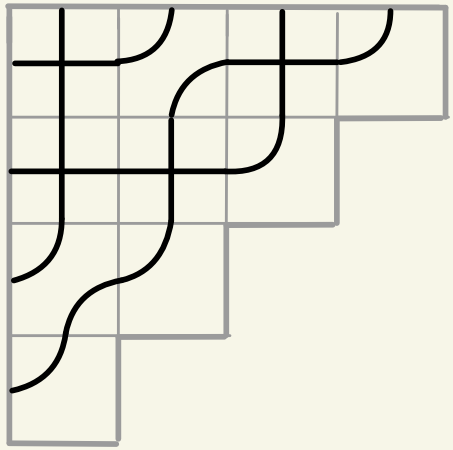
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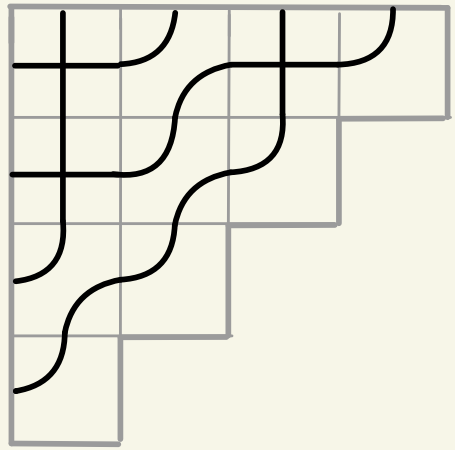


Pipe Dreams

PD(2413) = set of pipe dreams of 2413

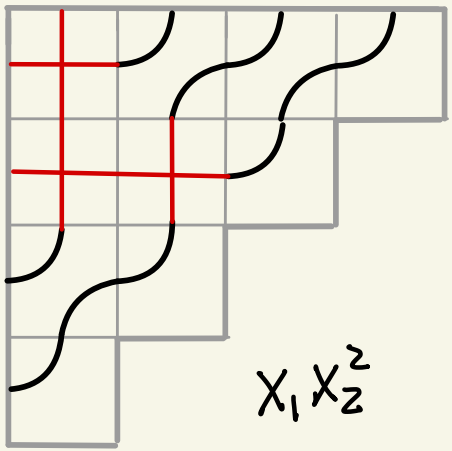




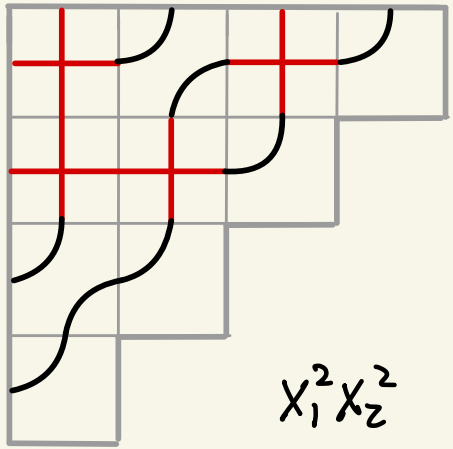


Pipe Dreams

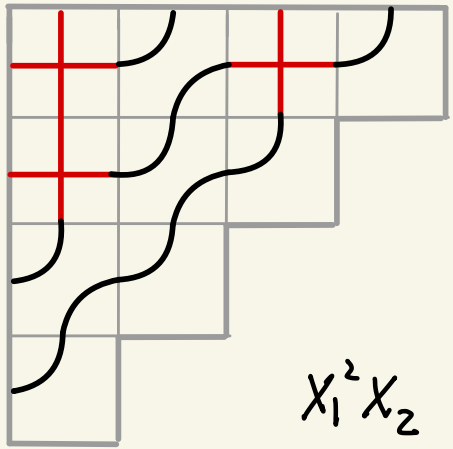
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$X_1 X_2^2$



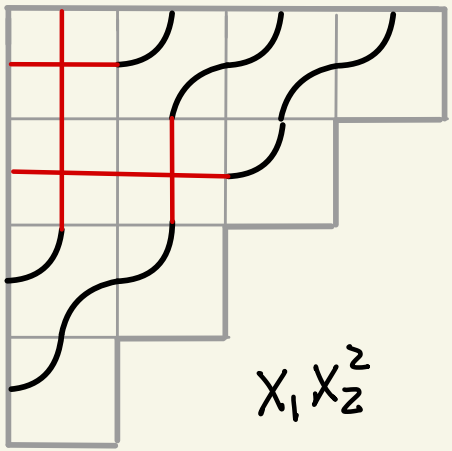
$X_1^2 X_2^2$



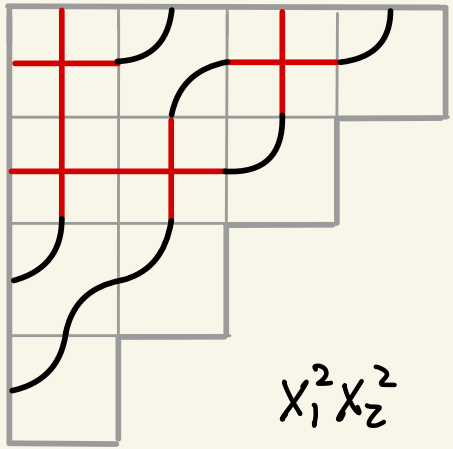
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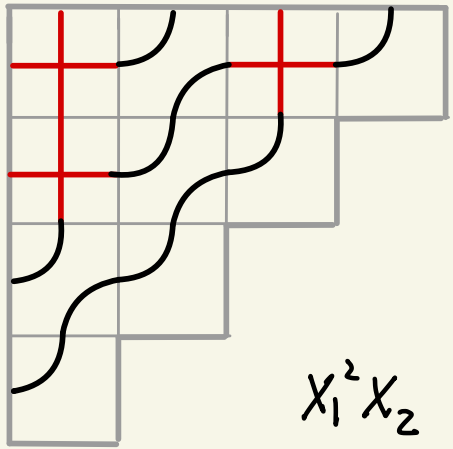
PD(2413) = set of pipe dreams of 2413



$X_1X_2^2$



$X_1^2X_2^2$



$X_1^2X_2$

$$G_{2413} = X_1^2X_2 + X_1X_2^2 - X_1^2X_2^2$$

Bumpless Vertical-less Pipe Dreams

Top Grothendieck

$\widehat{Gw}$ = highest degree terms of Gw

Top Grothendieck

$\widehat{G_w}$ = highest degree terms of G_w

Theorem [PSW21]

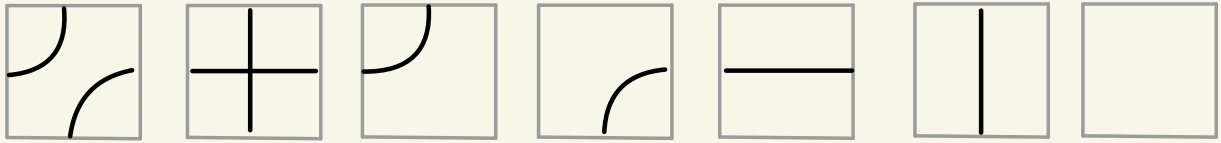
$\forall w \in S_n$. $\exists!$ inverse fireworks permutation u such that

$$\widehat{G_w} = \lambda_w \cdot \widehat{G_u}$$

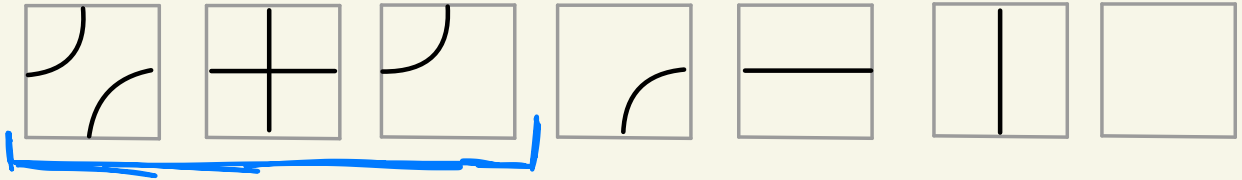
$$\lambda_w = \widehat{G_v}(1, 1, \cdots, 1)$$

v inverse fireworks

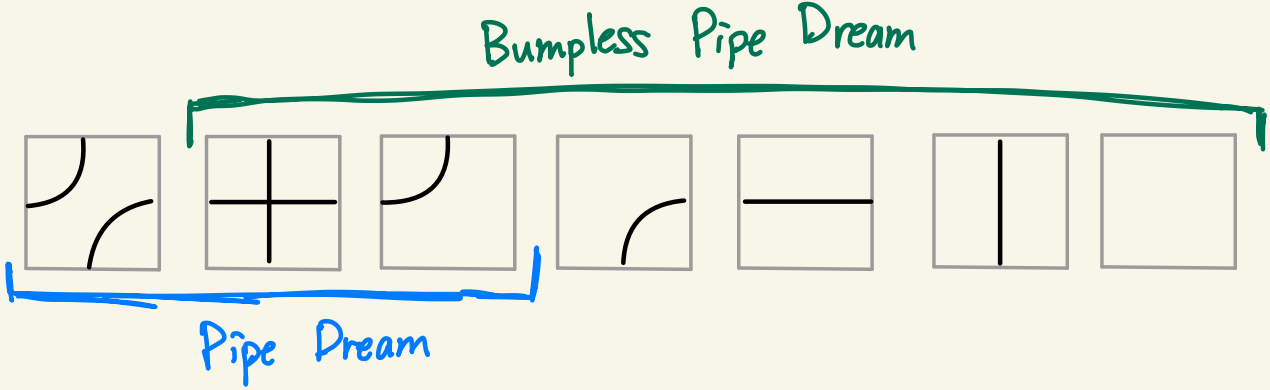
BVPD



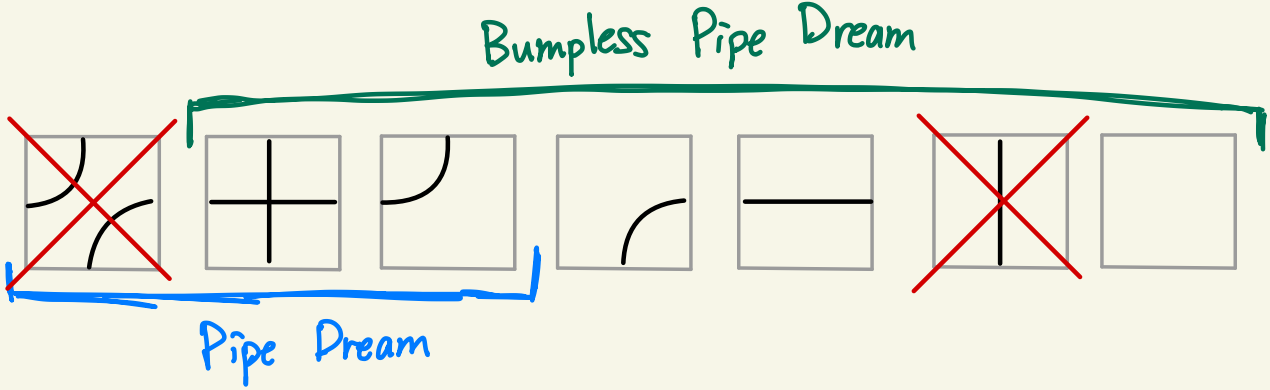
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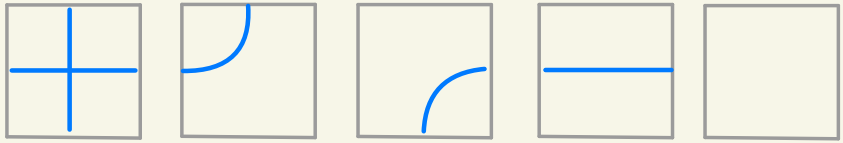


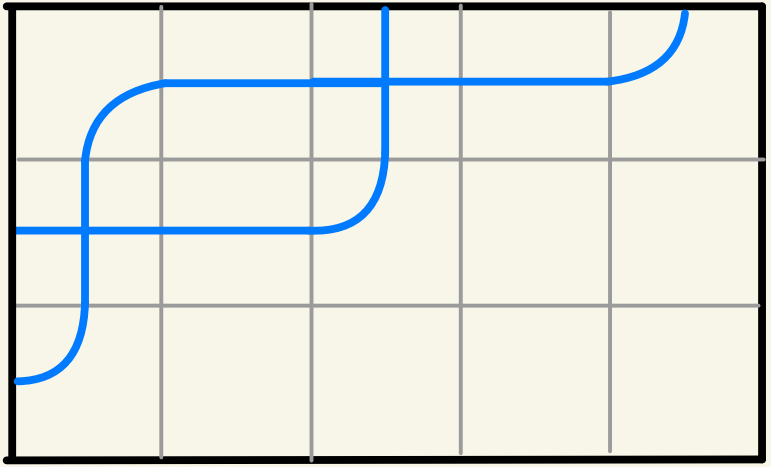
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Bumpless Vertical-less Pipe Dream

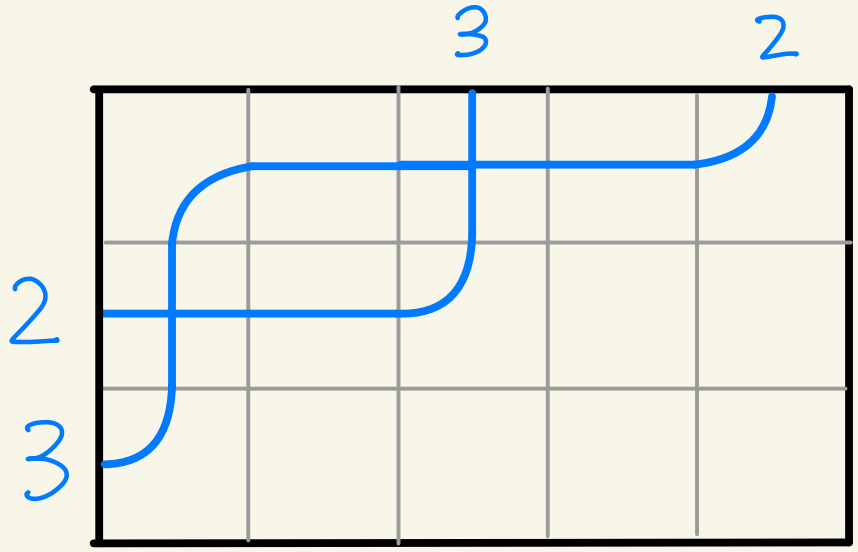
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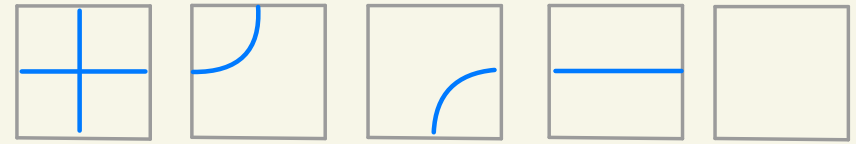


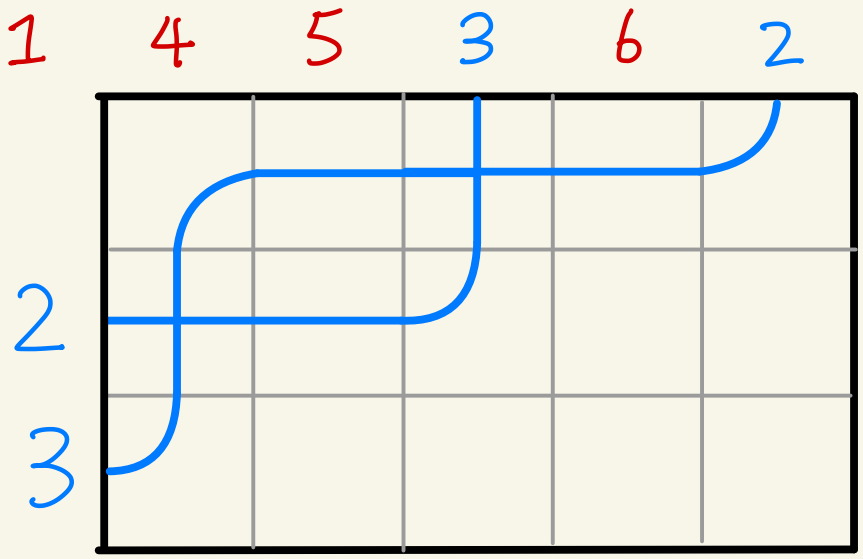
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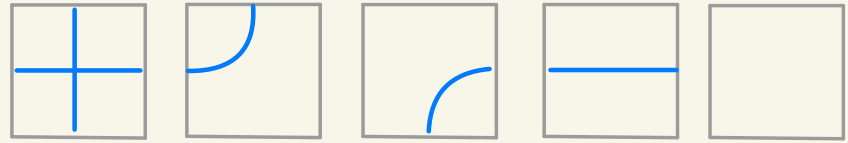


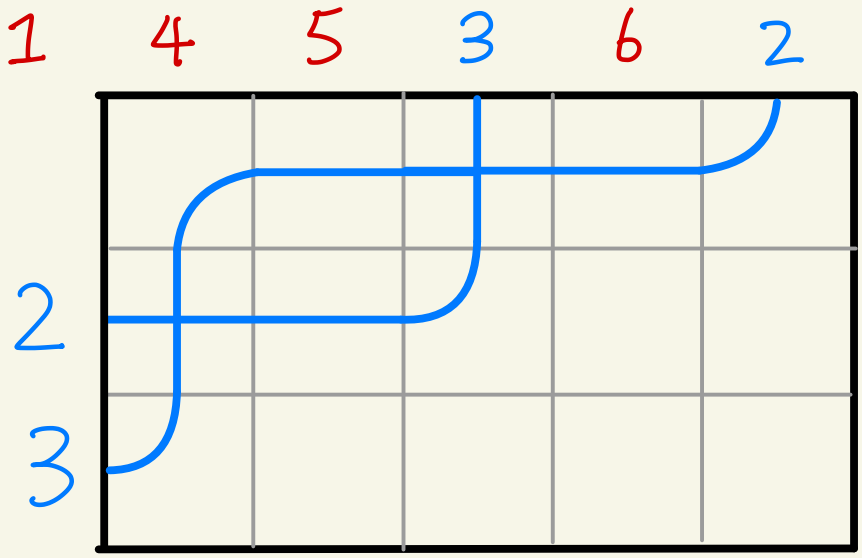
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BVPD





$w = 164235$

BVPD

Theorem [CY24]

For w inverse fireworks permutations,

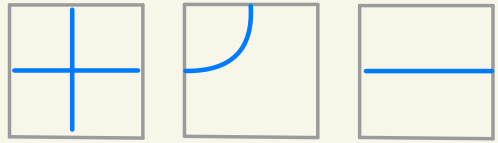
$$\hat{G}_w = \sum_{D \in \mathrm{BVPD}(w)} X^{\mathrm{wt}(D)}$$

BVPD

Theorem [CY24]

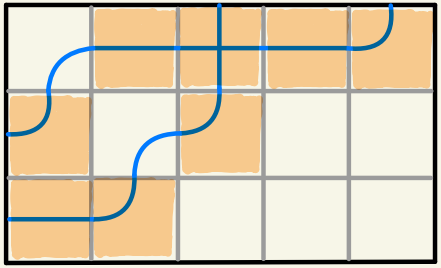
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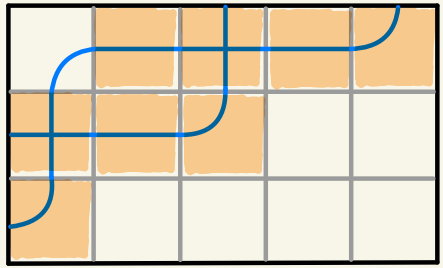
$\mathrm{wt}(D)$ = # of  in each row

BVPD

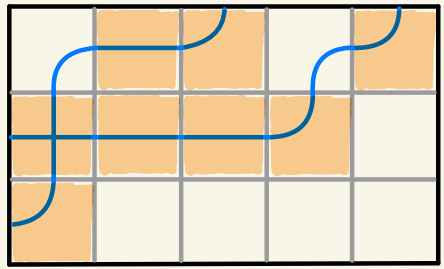
$x_1^4 x_2^2 x_3^2$



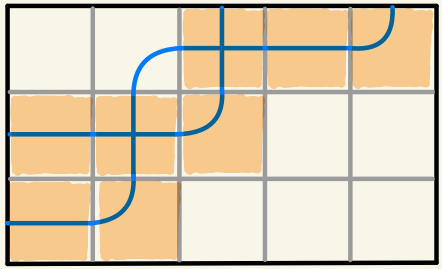
$x_1^4 x_2^3 x_3^1$

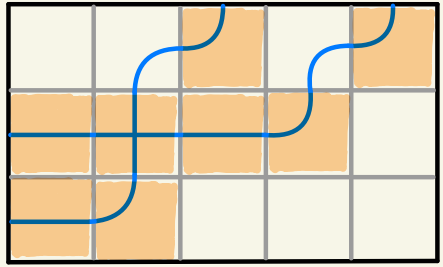


$x_1^3 x_2^4 x_3^1$



$x_1^3 x_2^3 x_3^2$

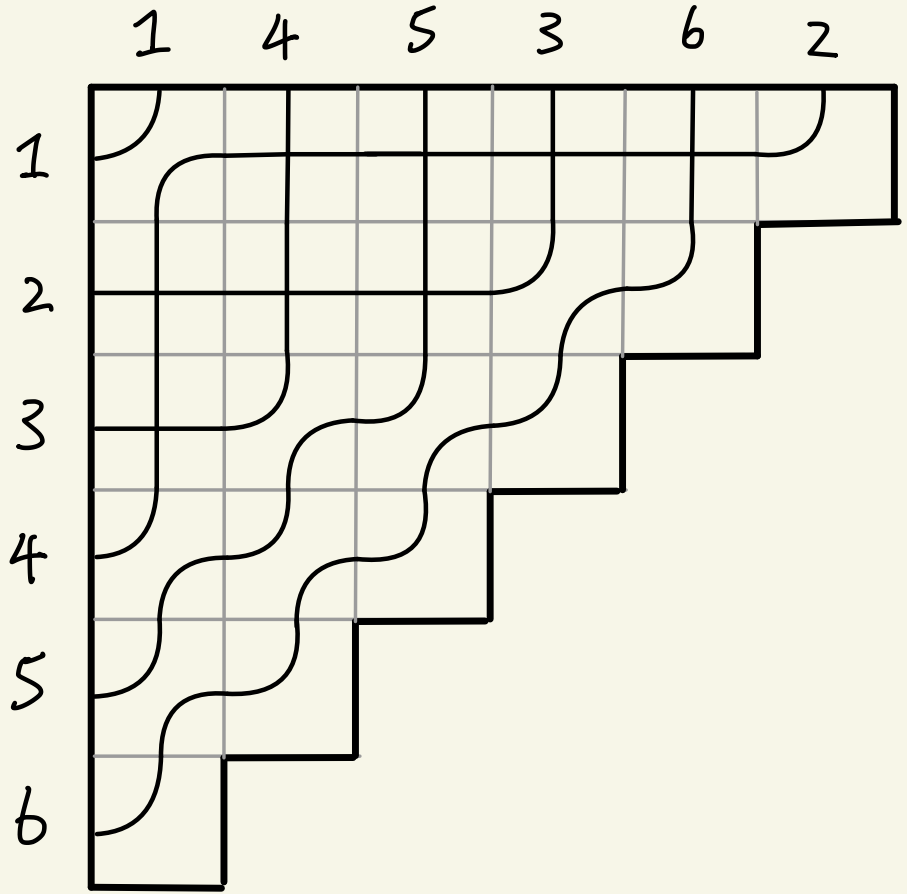




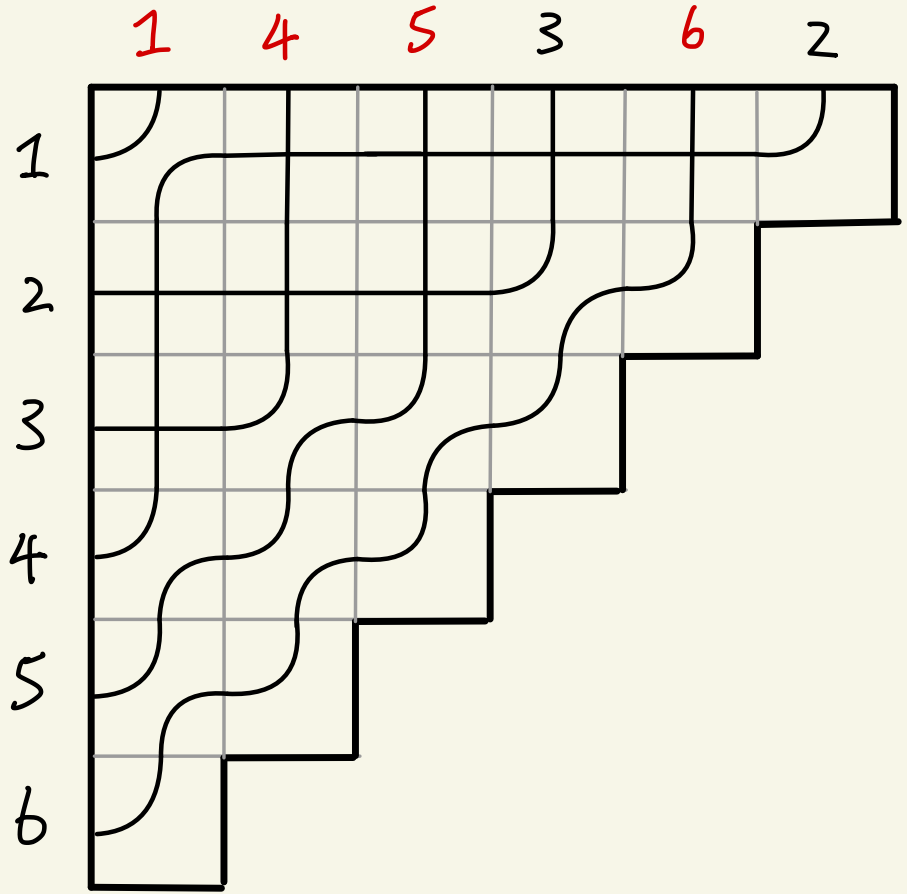
$x_1^2 x_2^4 x_3^2$

$$\widehat{G_W}(X) = x_1^4 x_2^2 x_3^2 + x_1^4 x_2^3 x_3^1 + x_1^3 x_2^4 x_3^1 + x_1^3 x_2^3 x_3^2 + x_1^2 x_2^4 x_3^2$$

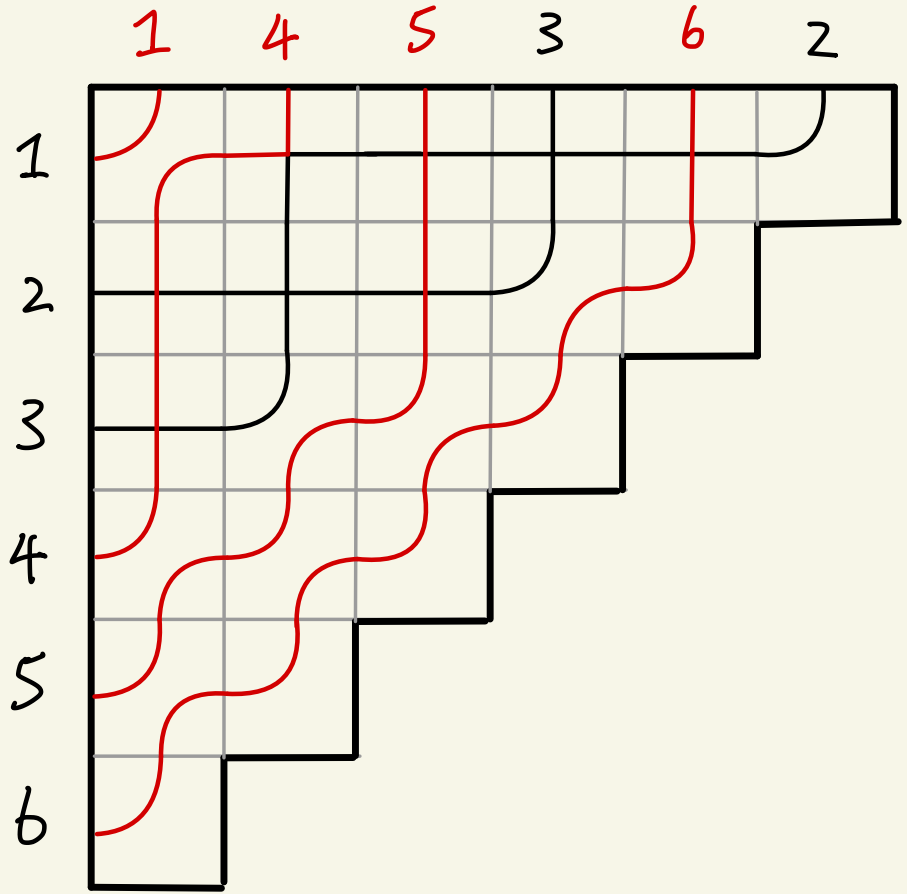
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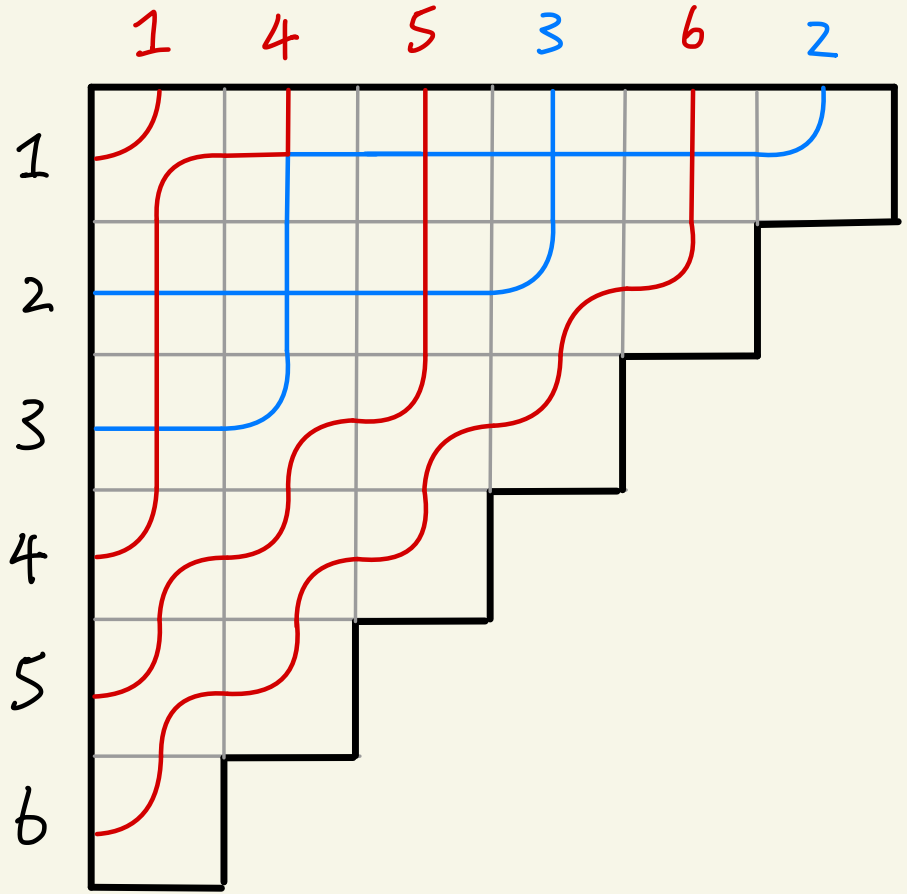
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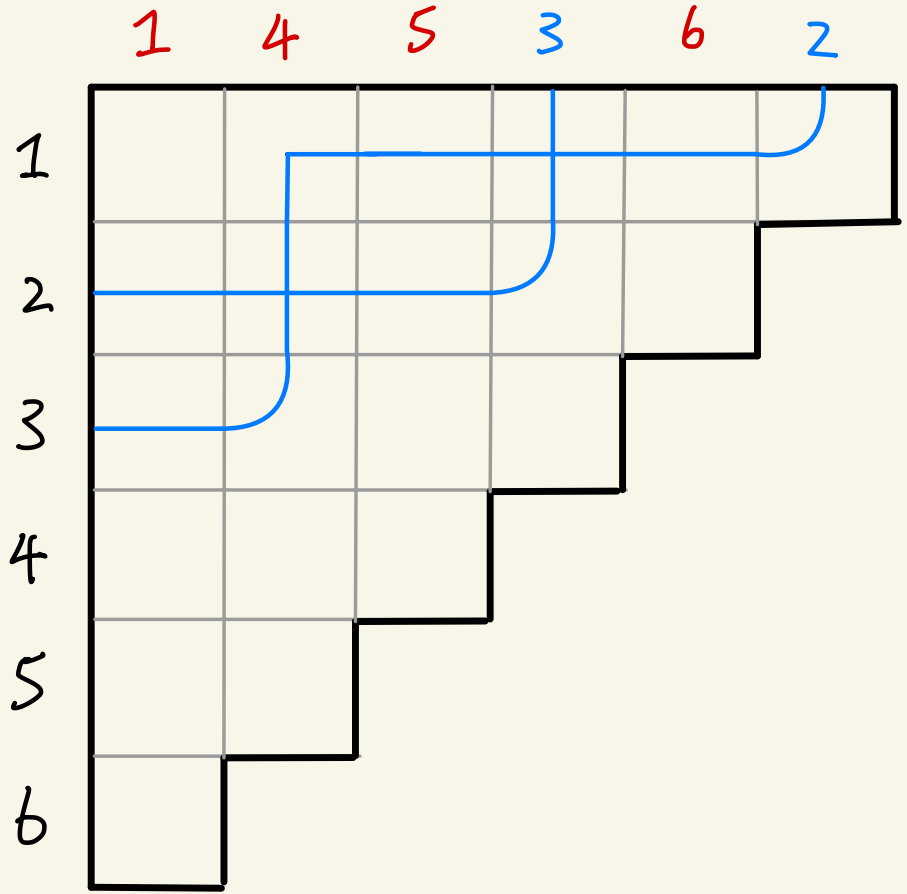
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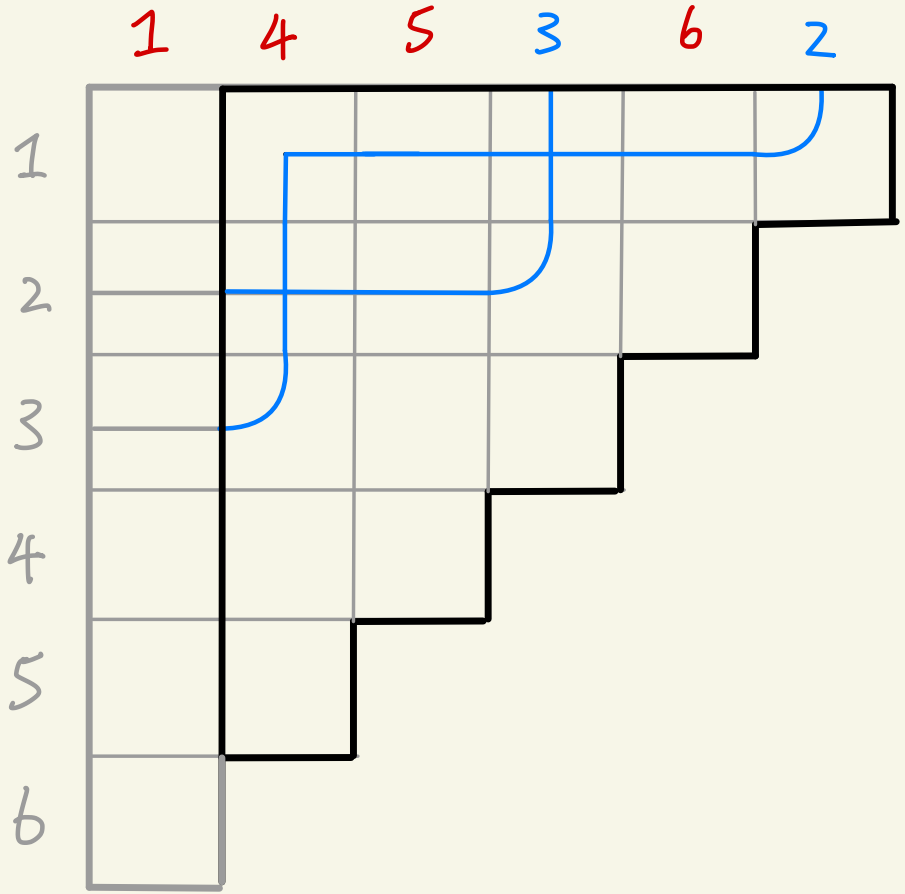
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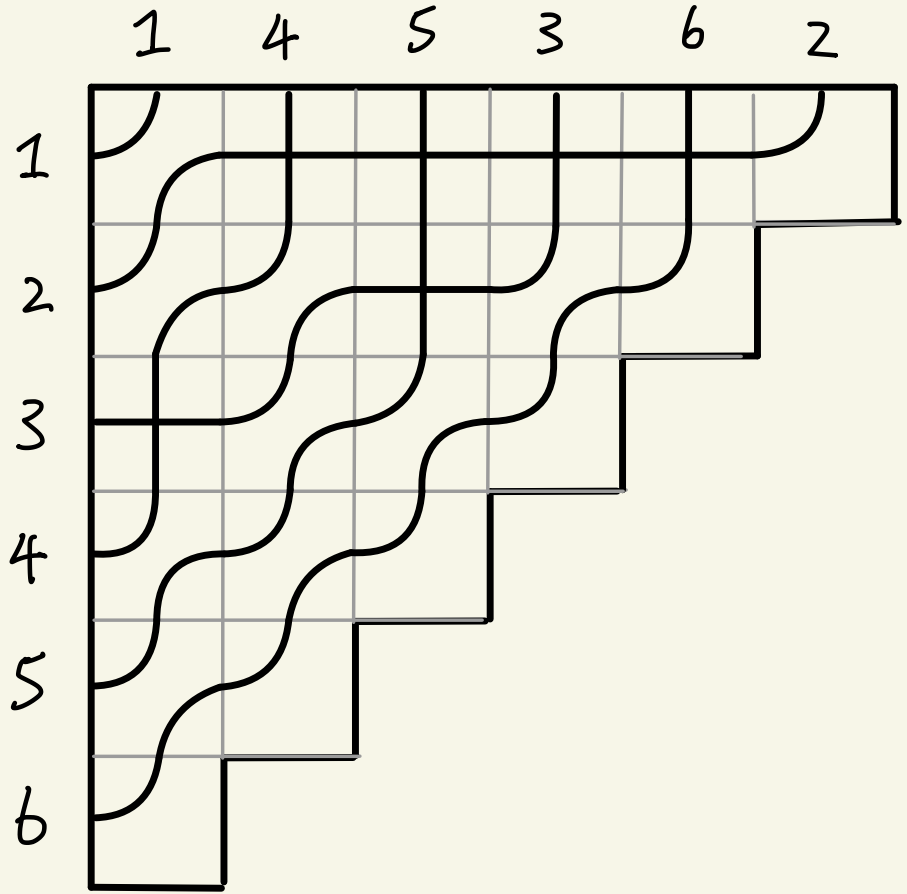
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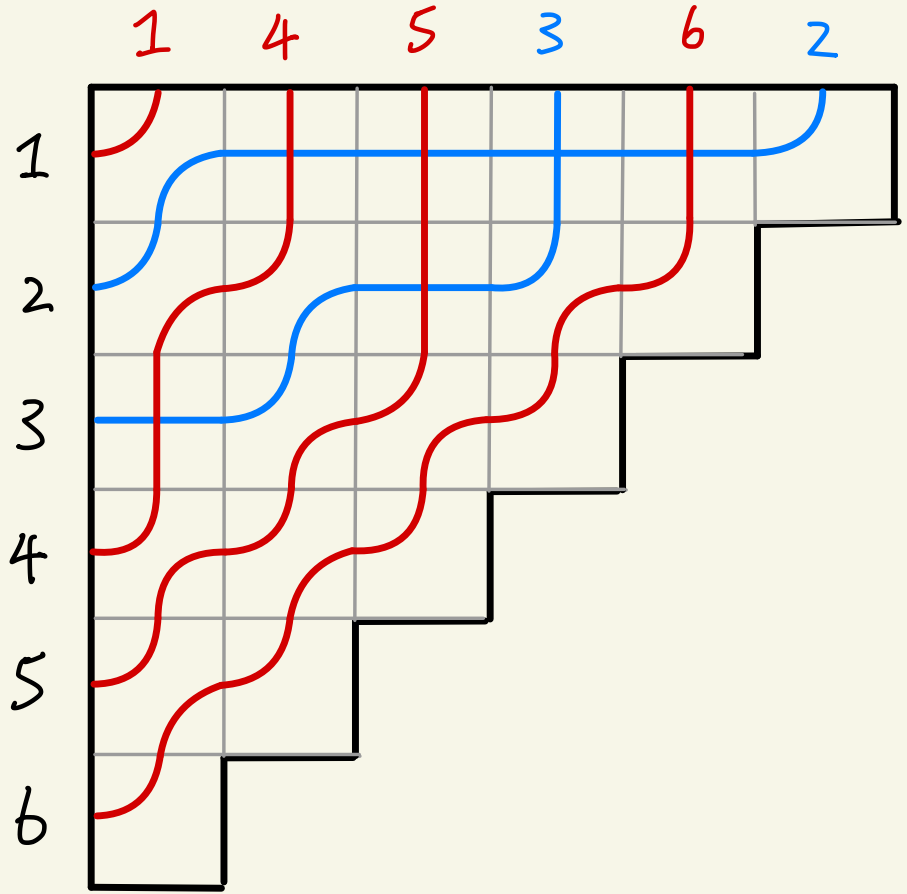
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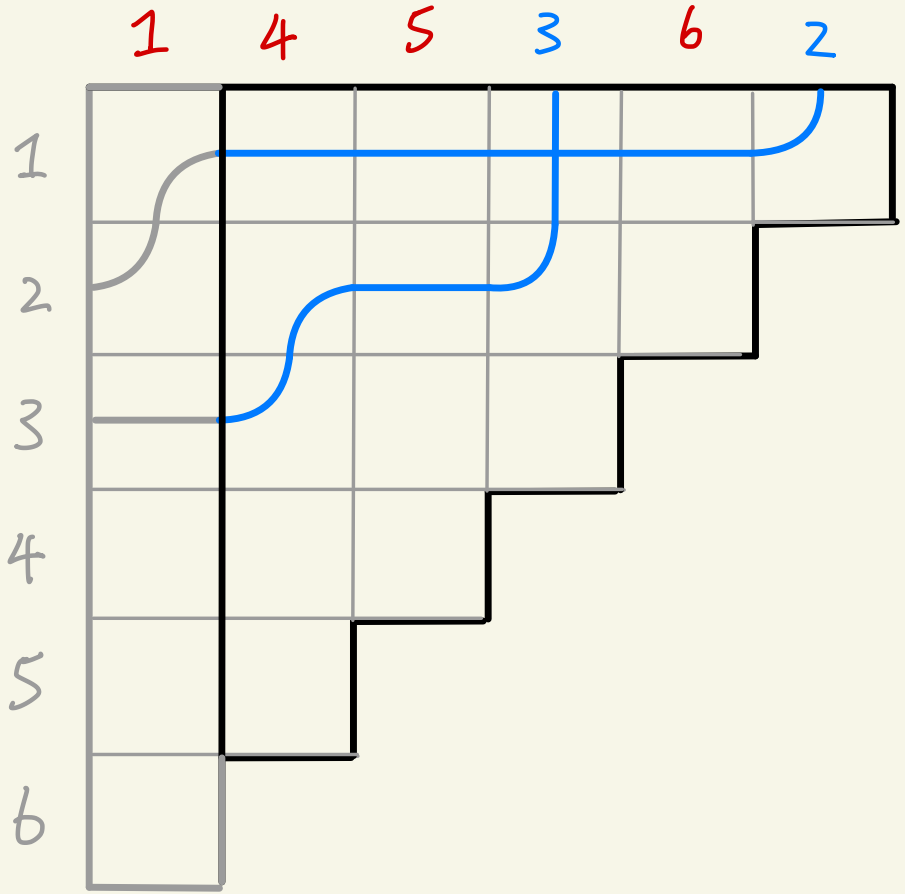
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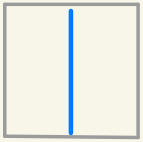


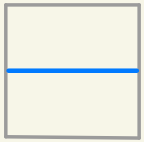
BVPD



BVPD

PD is heighest weight if and only if

1) Blue pipes have no  or 

2) Blue pipes in first column are 

Going-Up for Inverse Fireworks

Support of G_w

$$\mathrm{Supp}(G_w) = \{\text{monomials in } G_w\}$$ ~~coefficients~~

$$m \in \mathrm{Supp}(G_w) \iff \exists \text{ PD of } w \text{ with weight } m$$

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Big Important Conjecture [HMMSD 22]:

Homogenized Grothendieck polynomials

are Lorentzian.

Support of G_w

$$\mathrm{Supp}(G_w) = \{\text{monomials in } G_w\}$$ ~~coefficients~~

$m \in \mathrm{Supp}(G_w) \iff \exists$ PD of w with weight m

Big Important Conjecture [HMMSD 22]:

~~Homogenized~~ Grothendieck polynomials

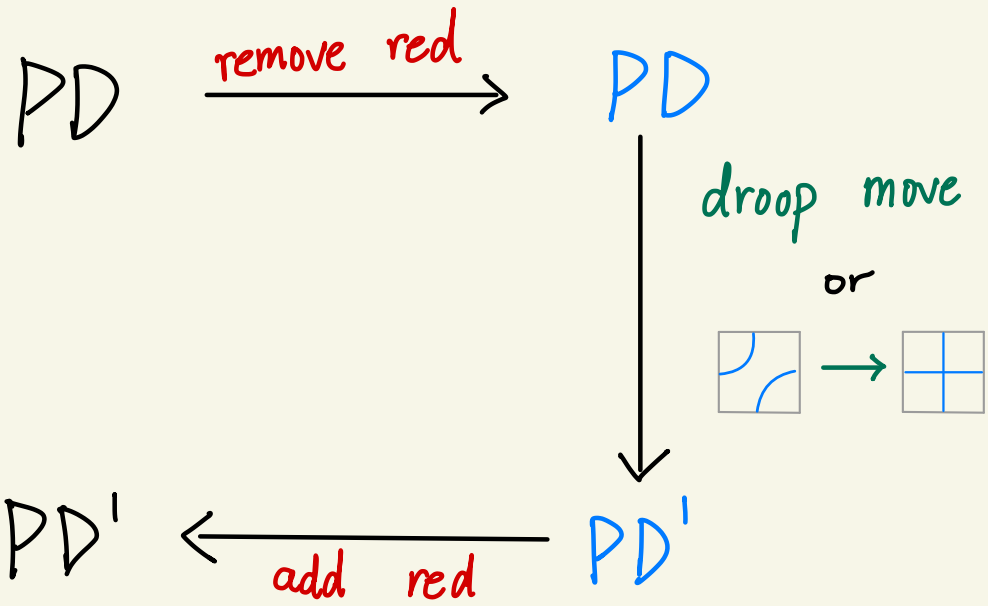
are ~~Lorentzian~~. "nice".

Support of G_w

Conjecture [MSSD 24] (Going Up Conjecture)

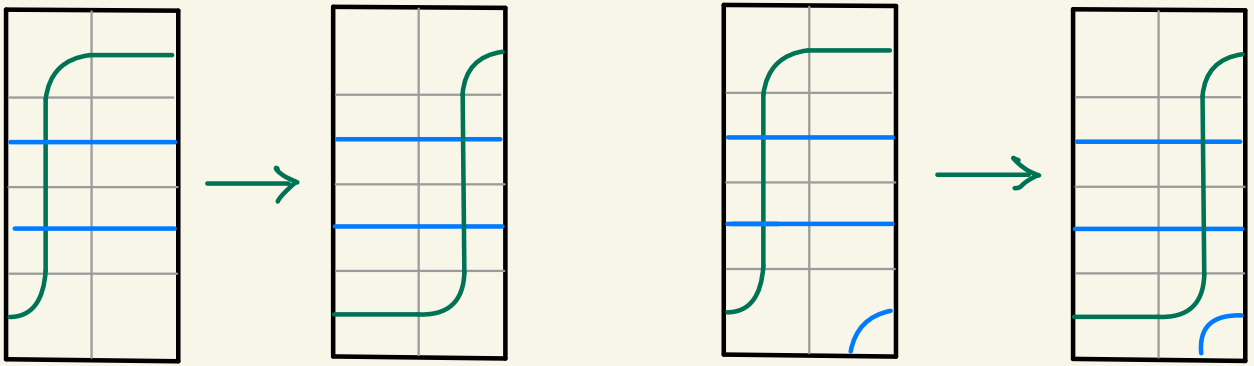
If $m \in \mathrm{Supp}(G_w)$ not highest degree, then $\exists\, x_i$ such that $x_i m \in \mathrm{Supp}(G_w)$.

Going Up

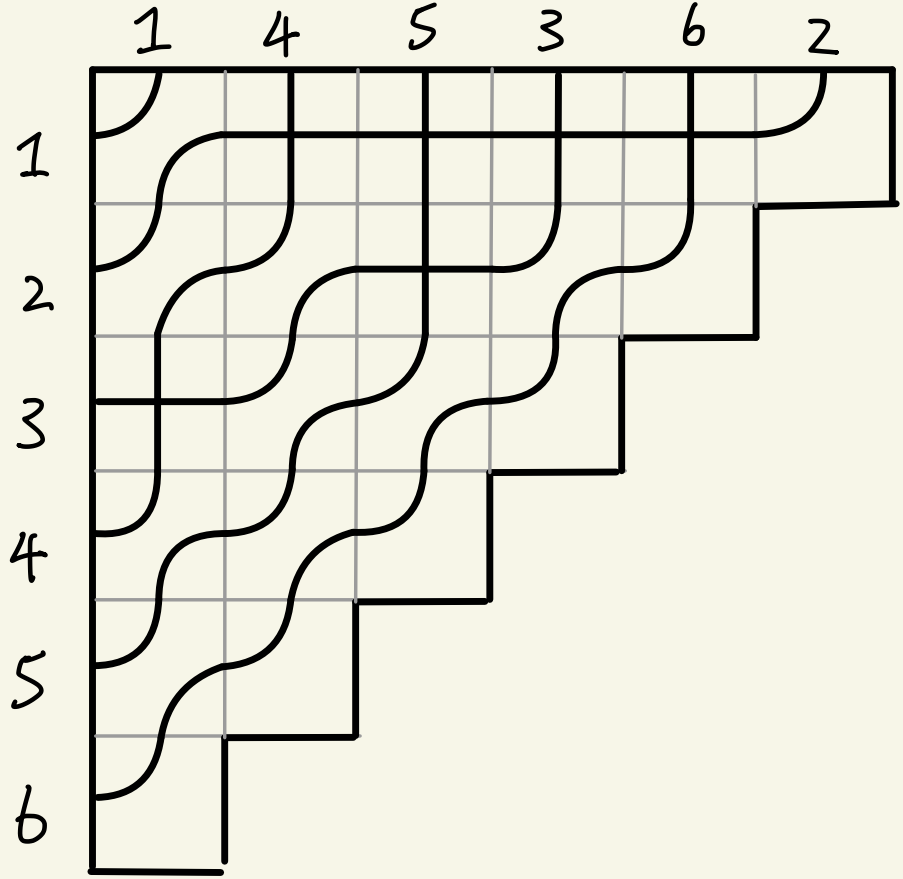


Going Up

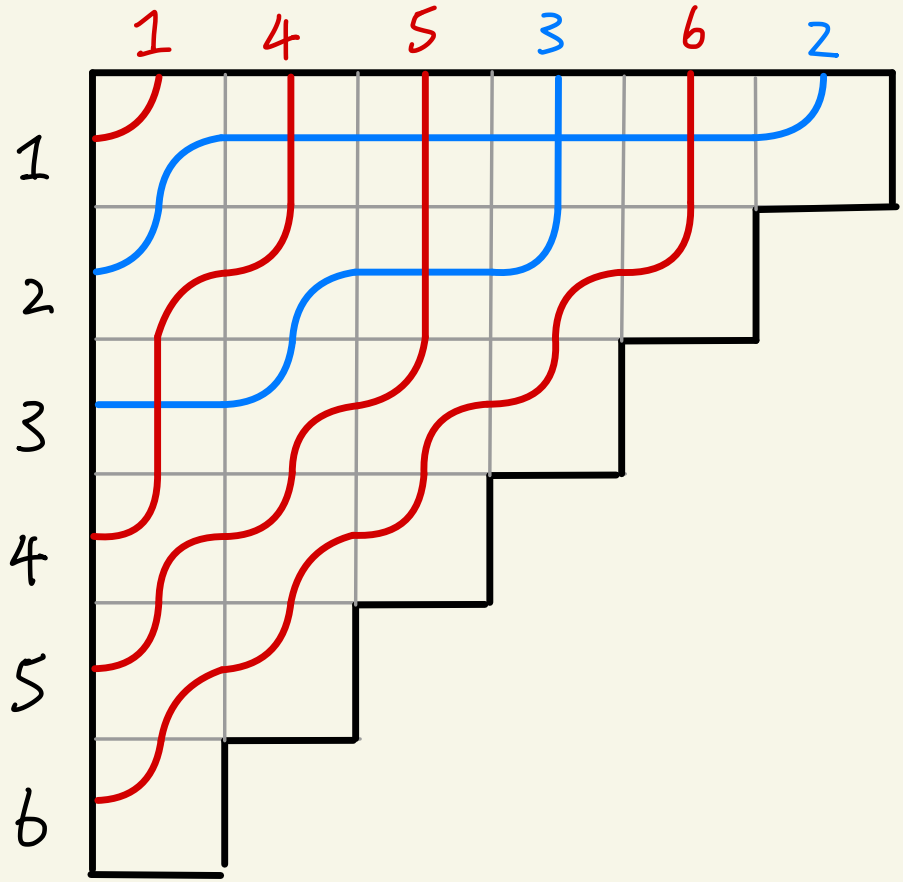
Droop moves



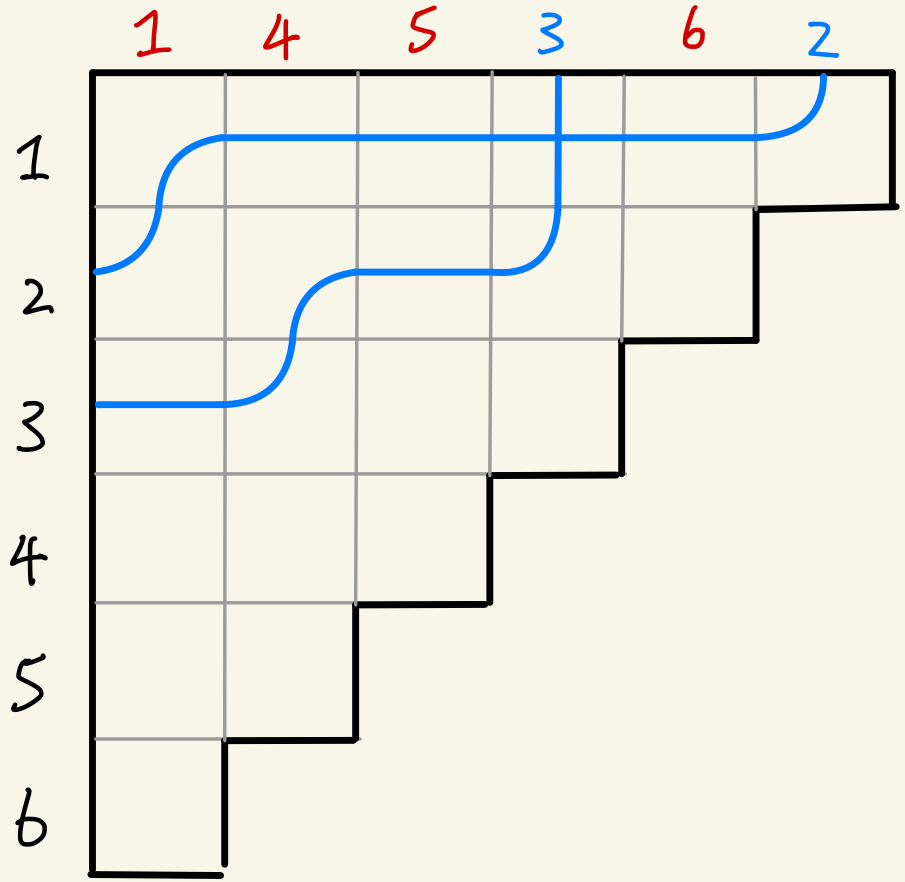
Going Up



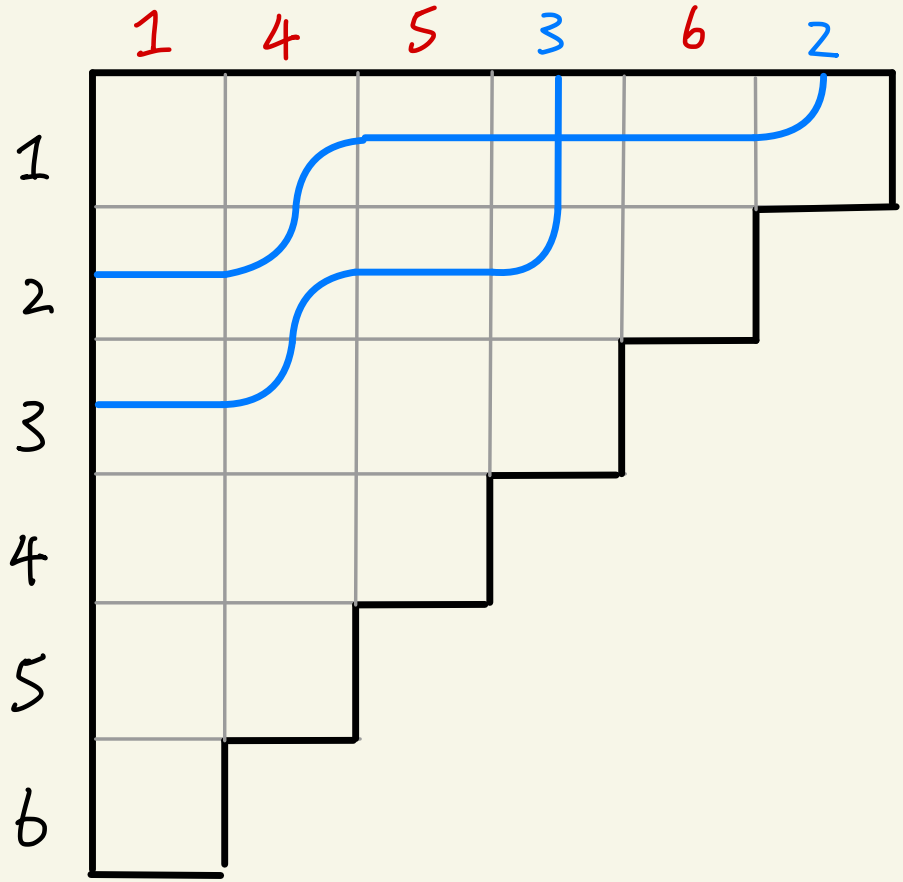
Going Up



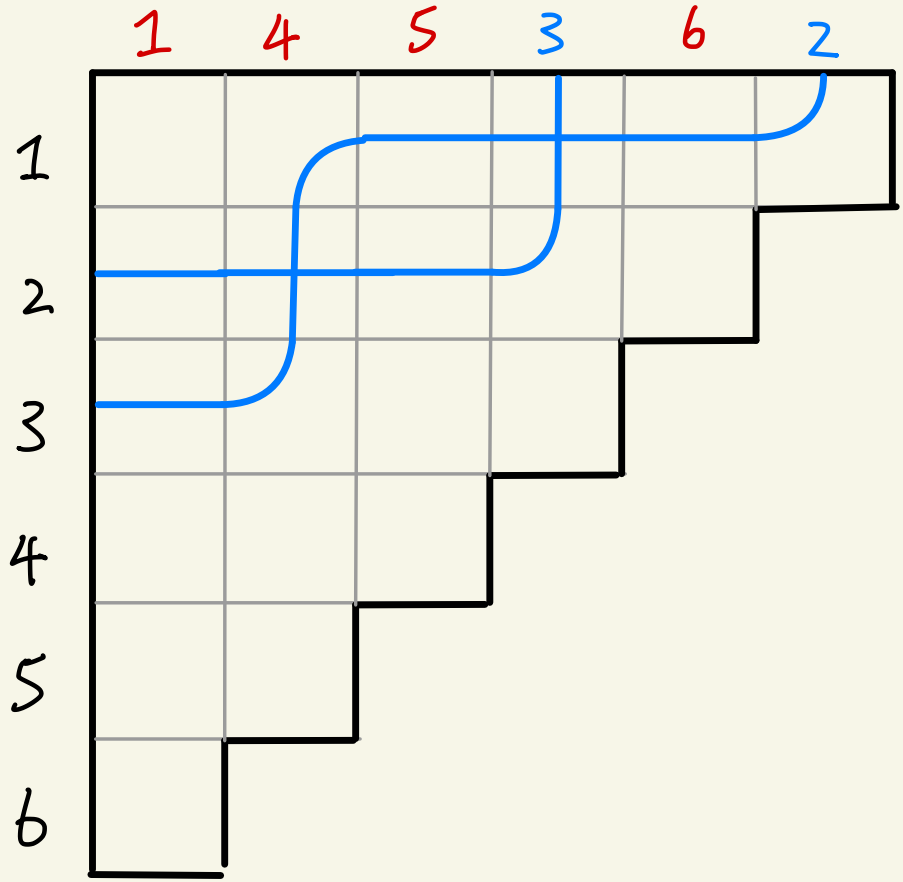
Going Up



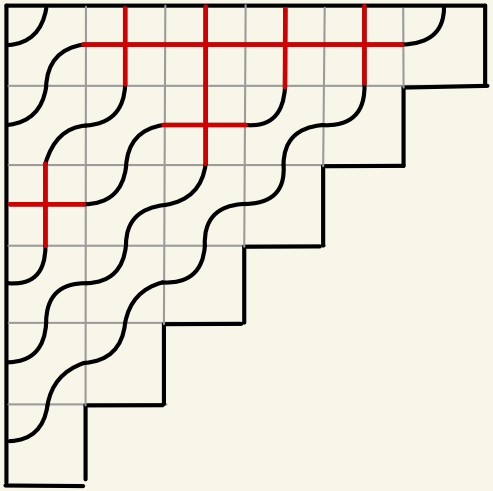
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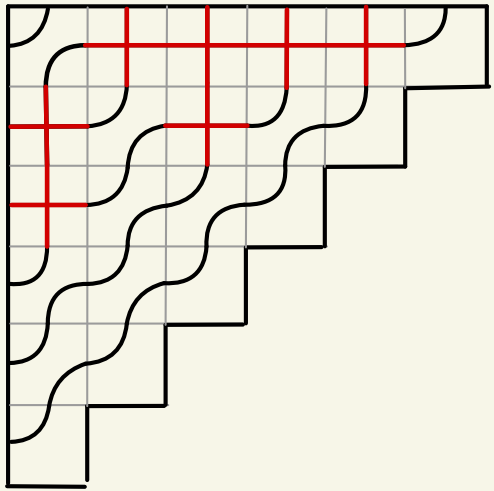


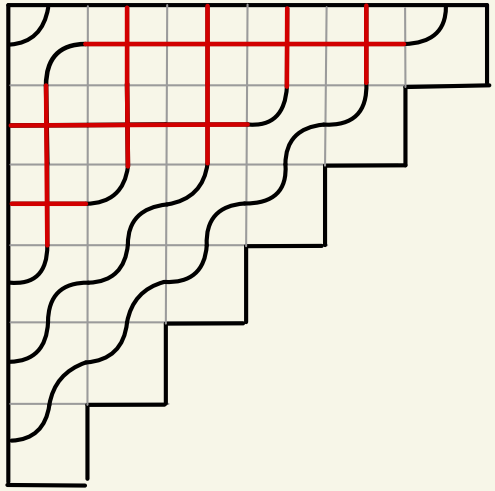
Going Up



Going Up







Going Up

If $m \in \mathrm{Supp}(G_w)$ not highest degree, then $\exists\, x_i$ such that $x_i m \in \mathrm{Supp}(G_w)$.

Thm [CY24]:

True for inverse fireworks permutations.



Thank You!



