Grothendieck Polynomials of Inverse Fireworks Permutations

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May 15th

UIUC Combinatorics Seminar

Outline

1. Inverse Fireworks Permutations
2. Grothendieck Polynomials (Pipe Dreams)
3. Bumpless Vertical-less Pipe Dreams
4. Going-Up for Inverse Fireworks
5. Double Grothendieck Polynomials

Inverse Fireworks

Fireworks: The decreasing runs are increasing.

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3 1 6 7 5 4 2

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3 < 7 ≮ 6

Inverse Fireworks

Why are they called

f*reworks?

Inverse Fireworks

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Inverse Fireworks

Fireworks ① 3__12 avoiding

Inverse Fireworks

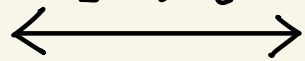
Fireworks ① 3__12 avoiding

② Bell number $B(n)$

Inverse Fireworks

Fireworks ① 3__12 avoiding

② Bell number B(n)

Fireworks  Inverse Fireworks

Grothendieck Polynomials and Pipe Dreams

Grothendieck Polynomials

K-theoretic analogues of Schubert polynomials

Grothendieck Polynomials

K-theoretic analogues of Schubert polynomials

$$G_w(x) = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = w_0 \\ \overline{\partial_i}\left(G_{ws_i}(x)\right) & \text{if } w(i) < w(i+1) \end{cases}$$

Grothendieck Polynomials

K-theoretic analogues of Schubert polynomials

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$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}}$$

$$\overline{\partial_i}(f) = \partial_i\left((1 - x_{i+1}) f\right)$$

Grothendieck Polynomials

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$$\overline{\partial_1}(x_1^2) =$$

Grothendieck Polynomials

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$$\overline{\partial_1}(x_1^2) = \partial_1(x_1^2 - x_1^2 x_2) = \frac{(x_1^2 - x_1^2 x_2) - (x_2^2 - x_1 x_2^2)}{x_1 - x_2}$$

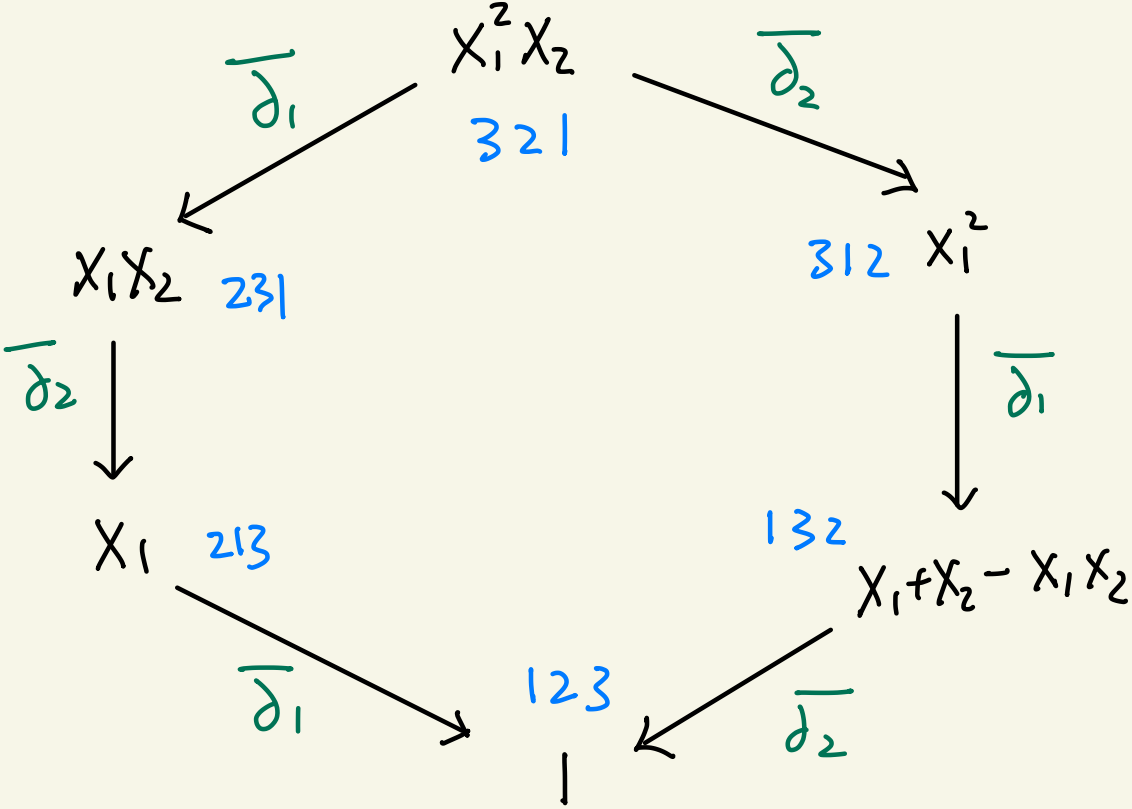
Grothendieck Polynomials

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$$= \frac{(x_1 + x_2)(x_1 - x_2) - x_1 x_2 (x_1 - x_2)}{x_1 - x_2} = x_1 + x_2 - x_1 x_2$$

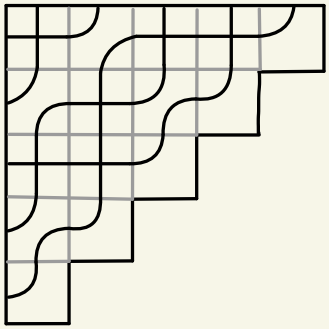
Grothendieck Polynomials

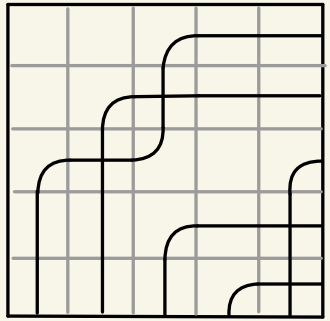


Grothendieck Polynomials

Combinatorial formulas:

Pipe Dreams and Bumpless Pipe Dreams



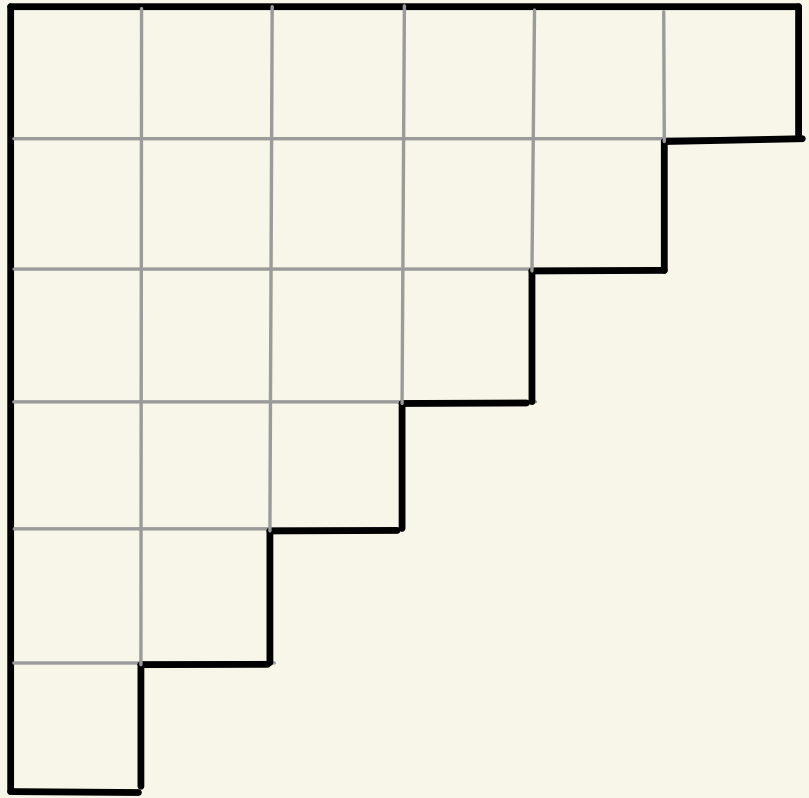


Pipe Dreams

Fill in tiles

Trace pipes
(Double Crossing)

Label sides

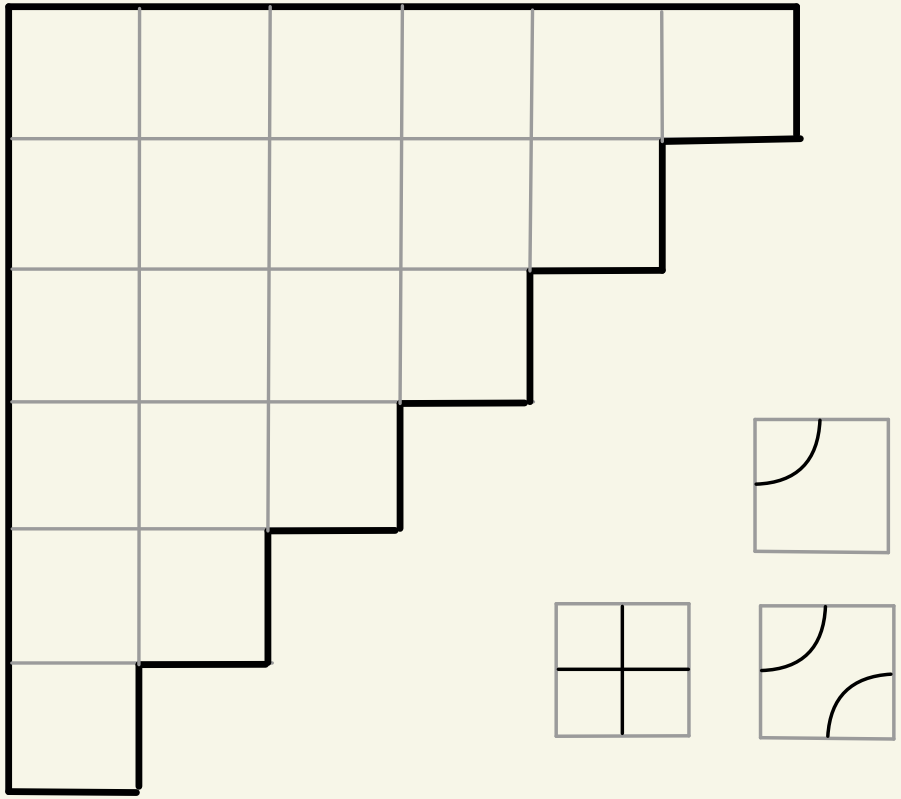


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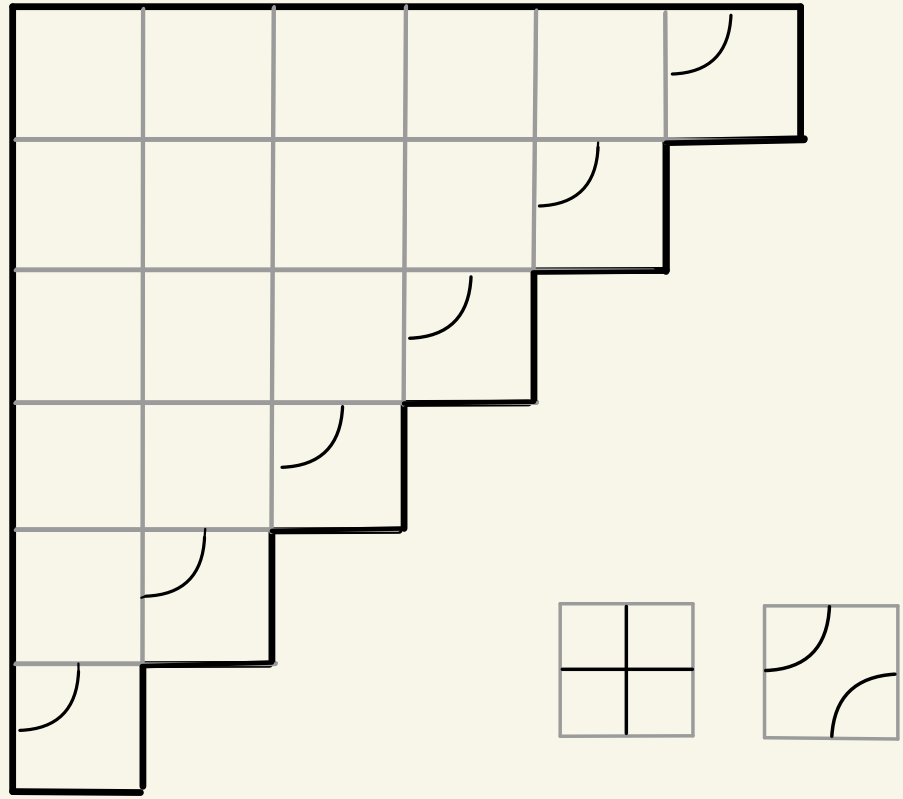


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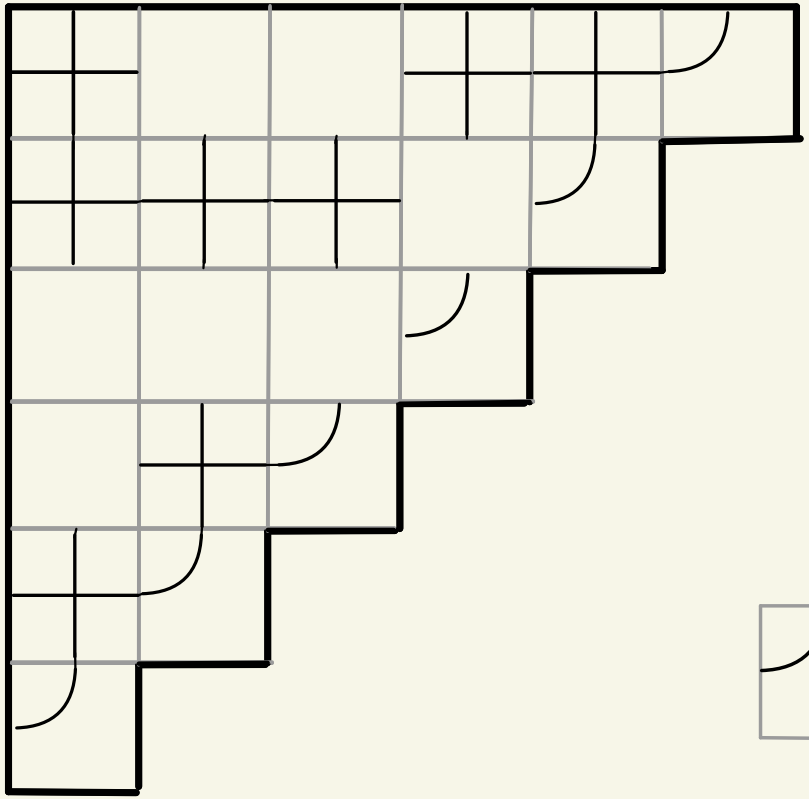


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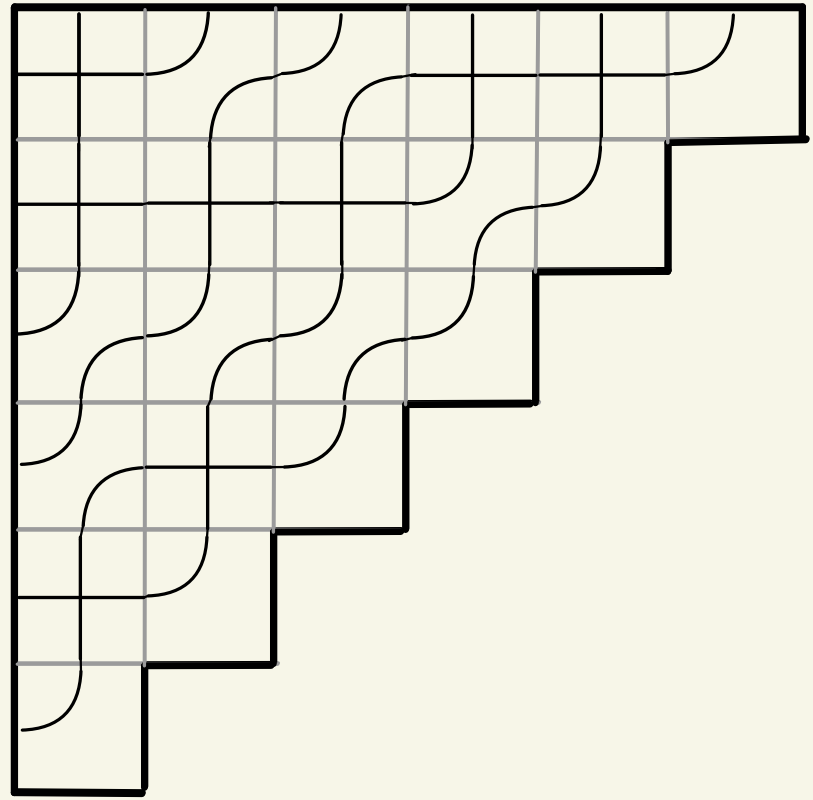


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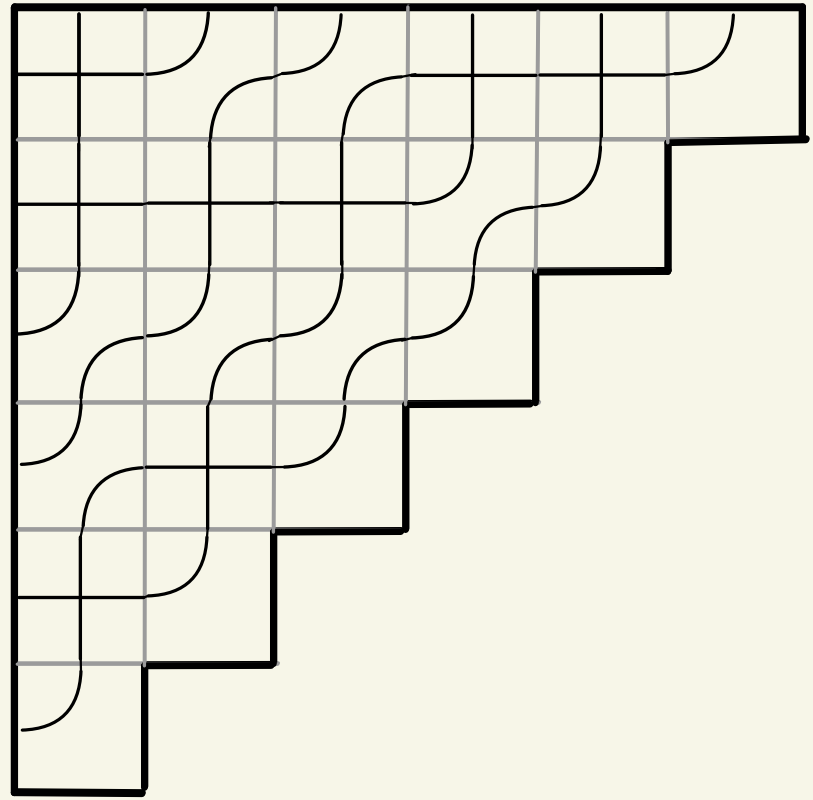


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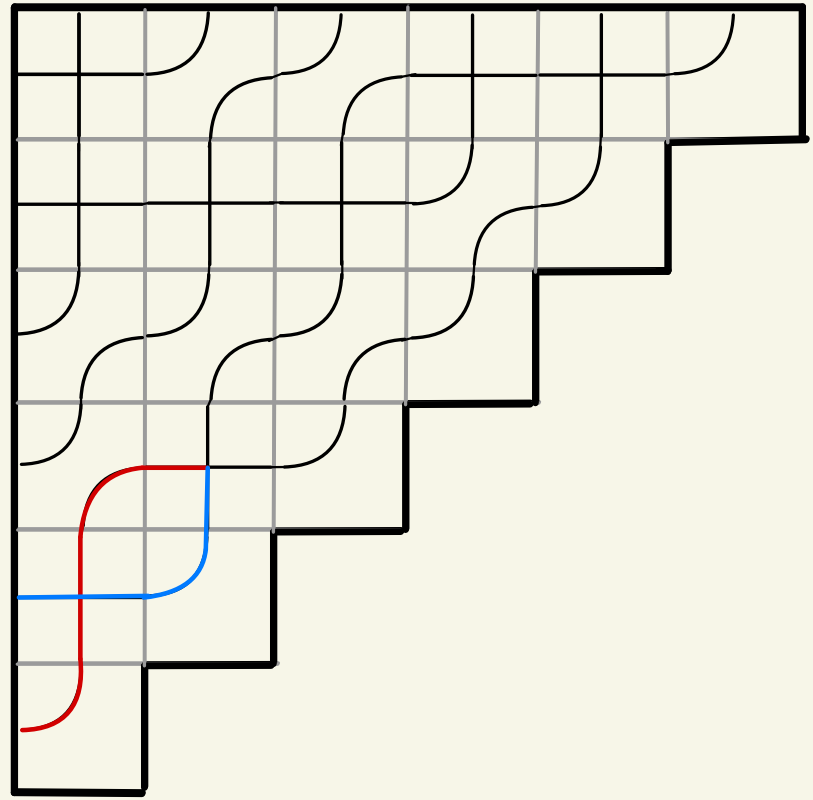


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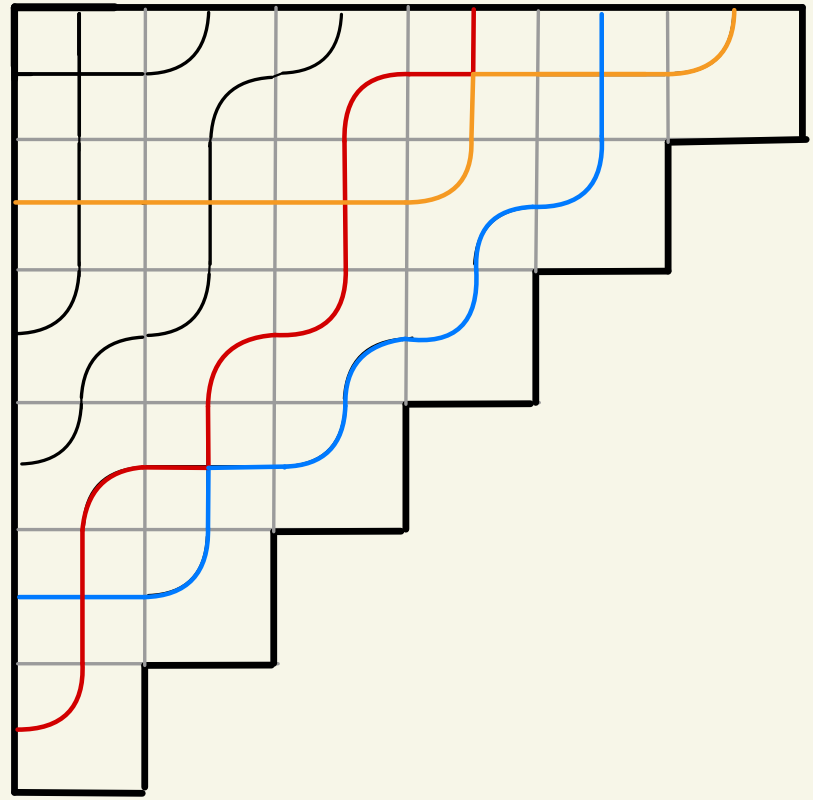


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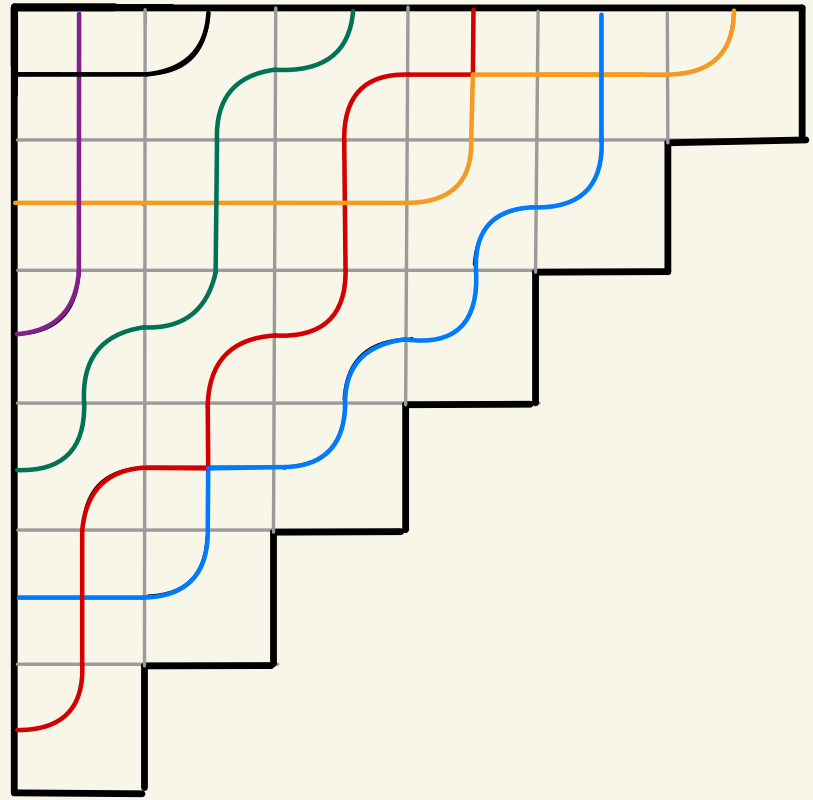


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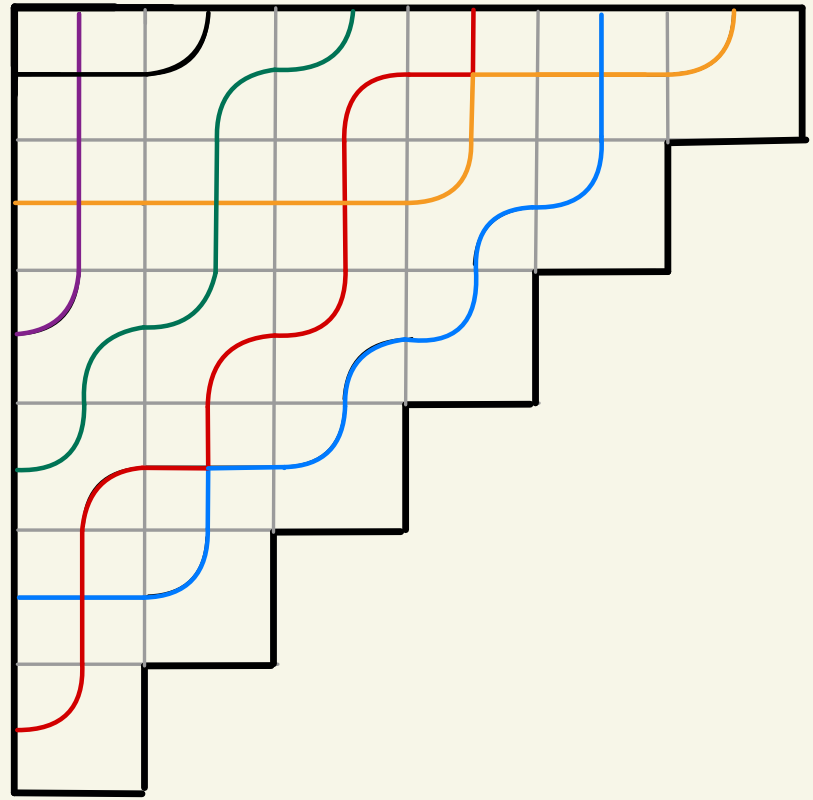


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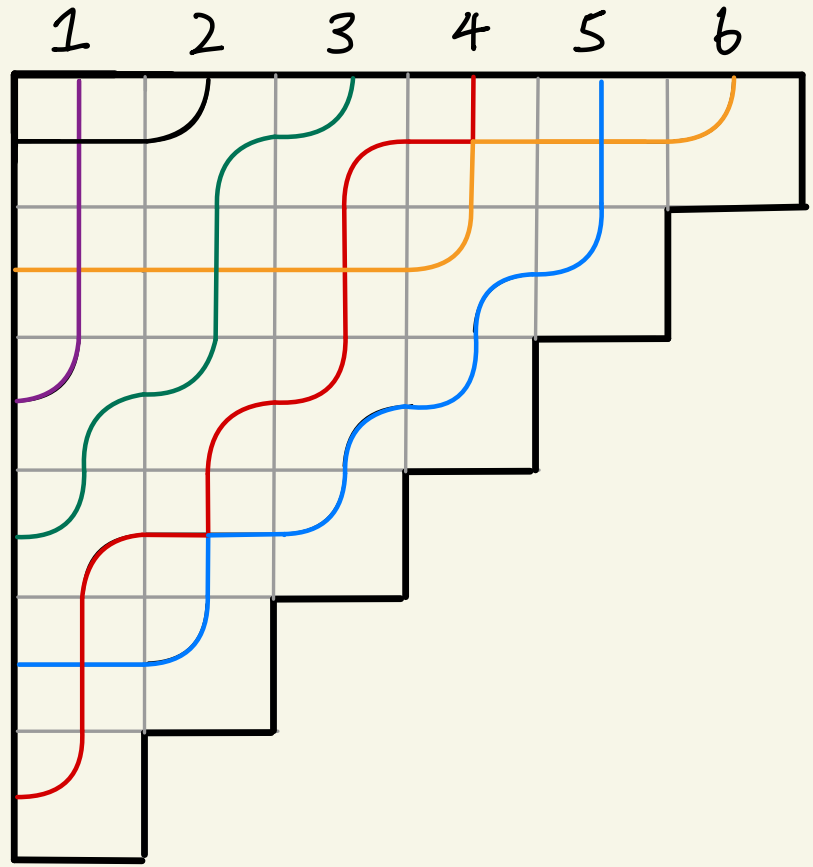


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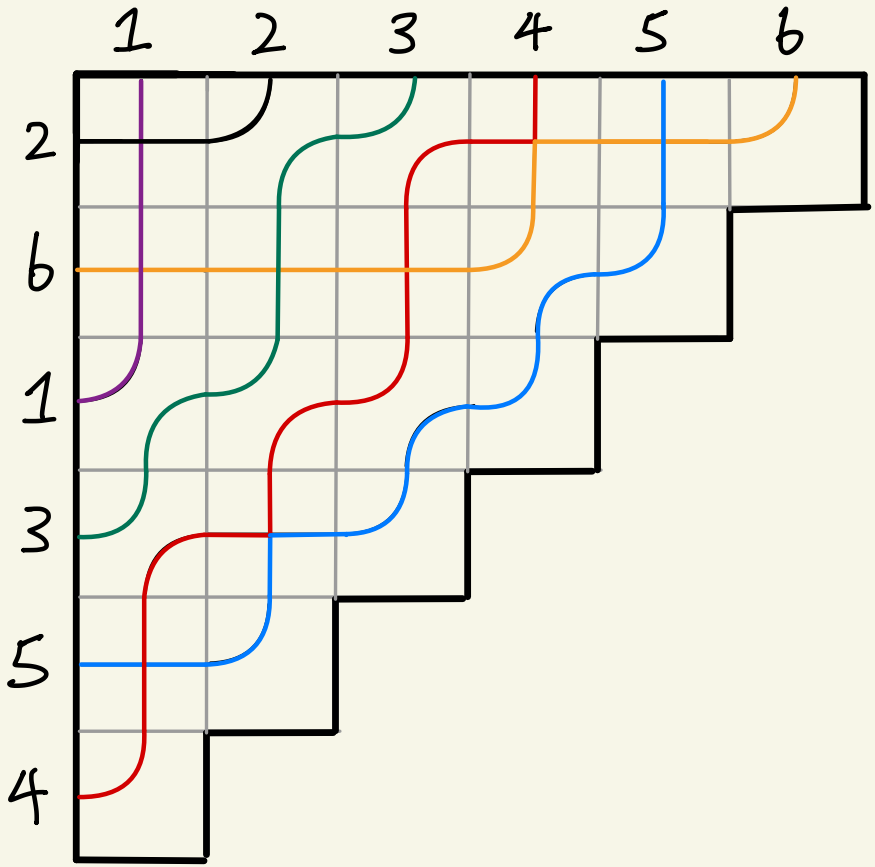


Pipe Dreams

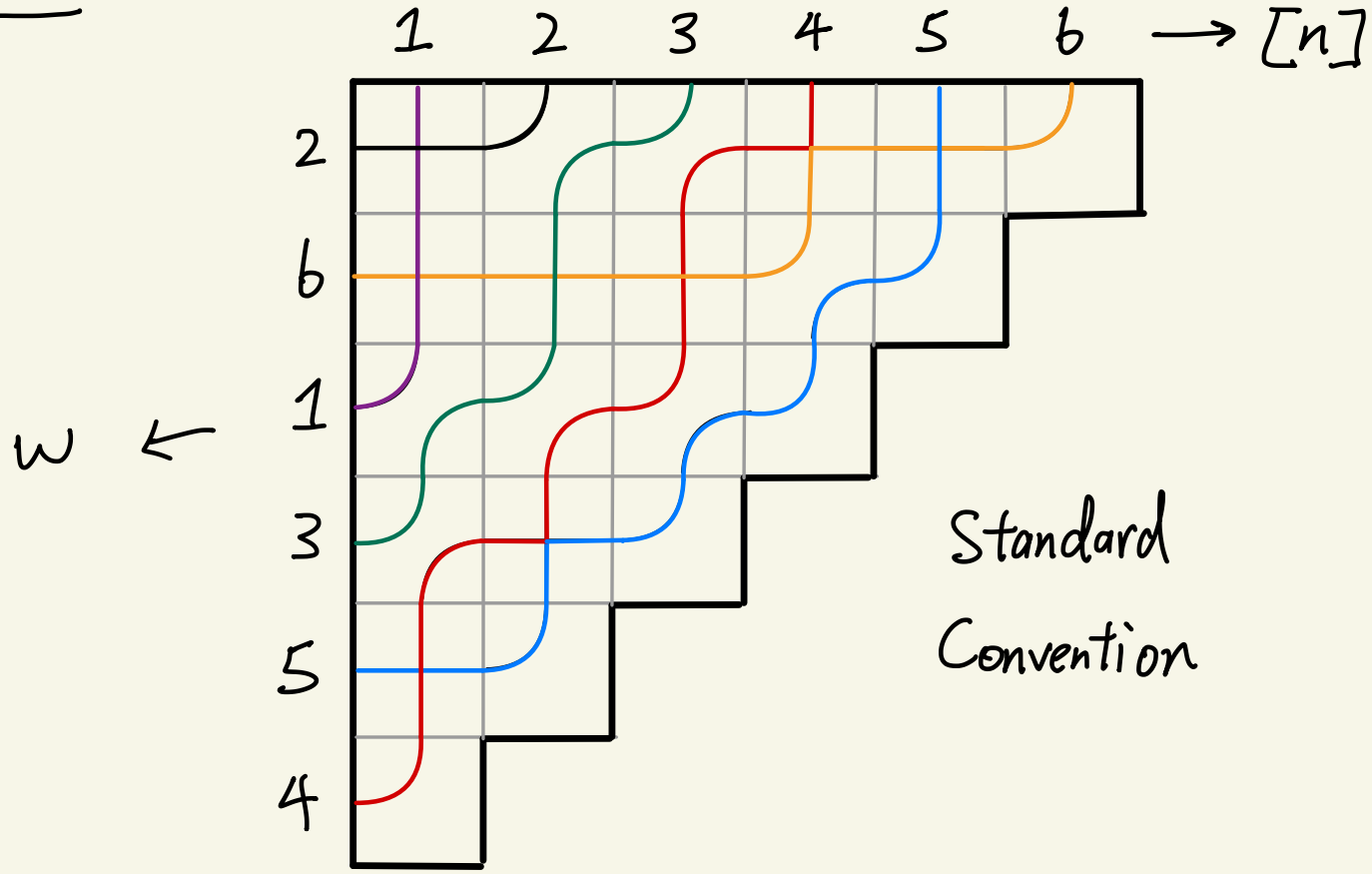
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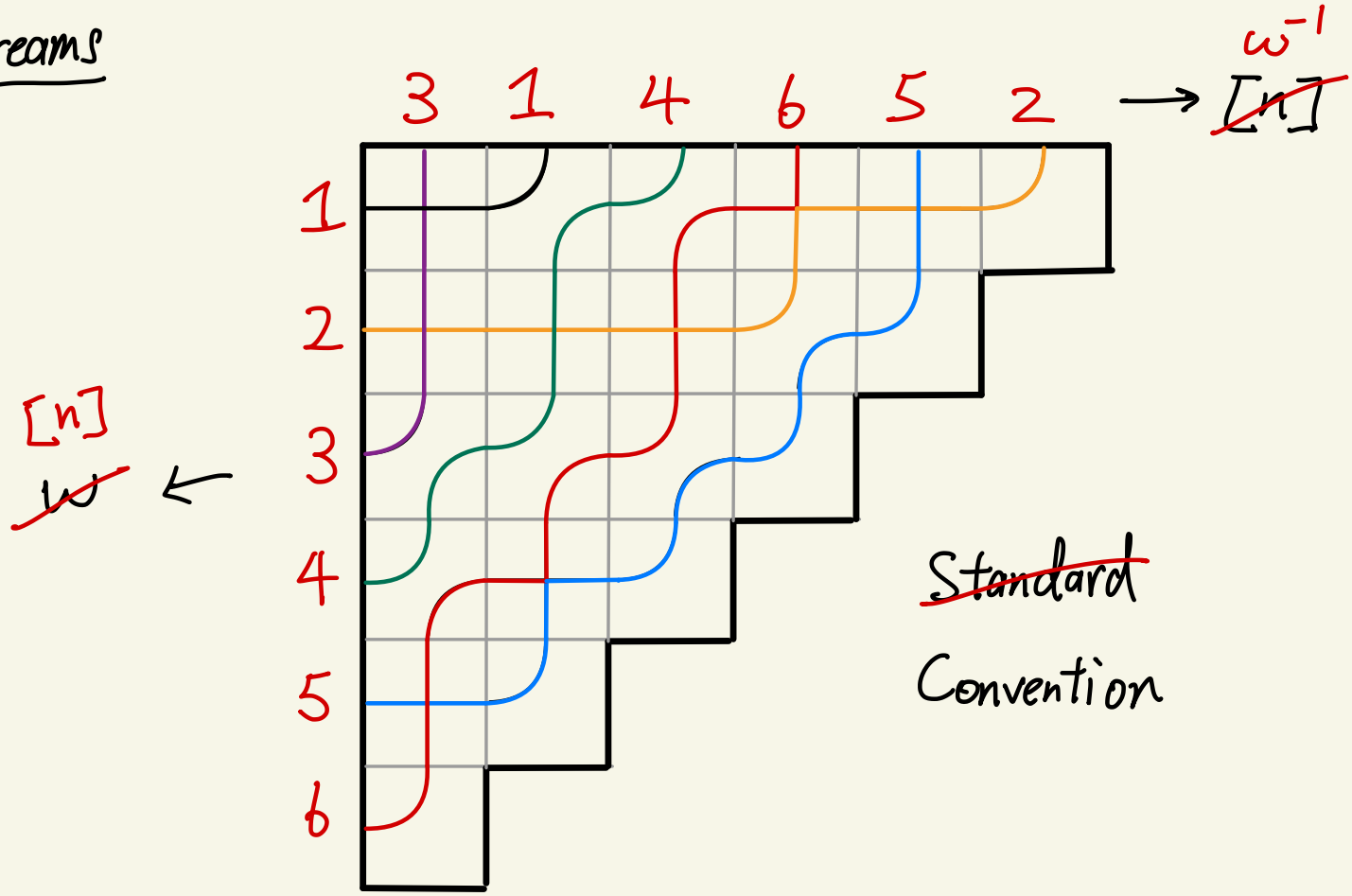
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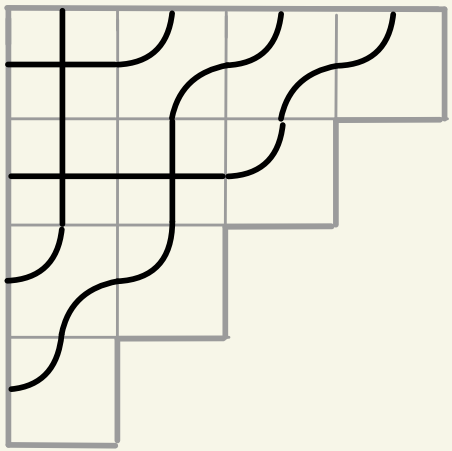
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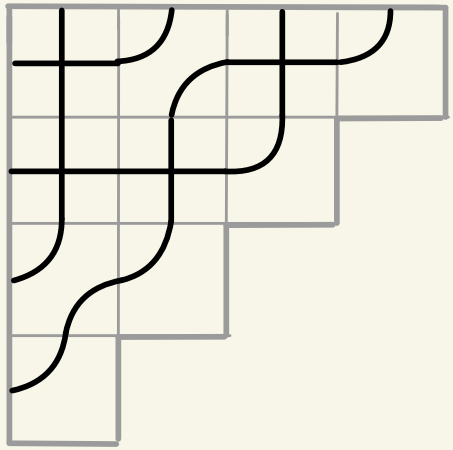


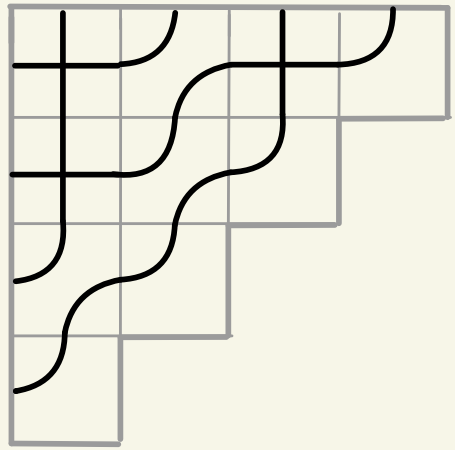
Pipe Dreams



Pipe Dreams PD(2413) = set of pipe dreams of 2413

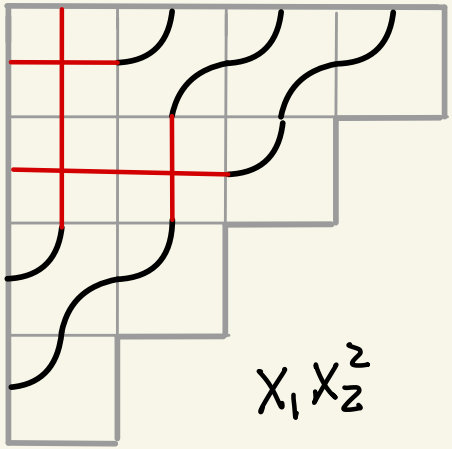




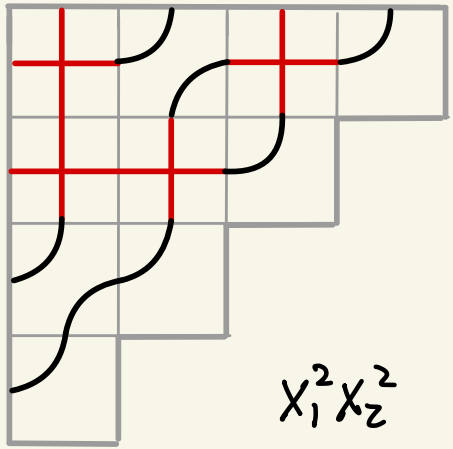


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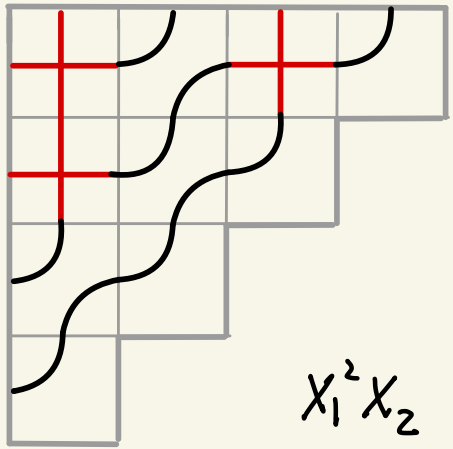
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$X_1 X_2^2$



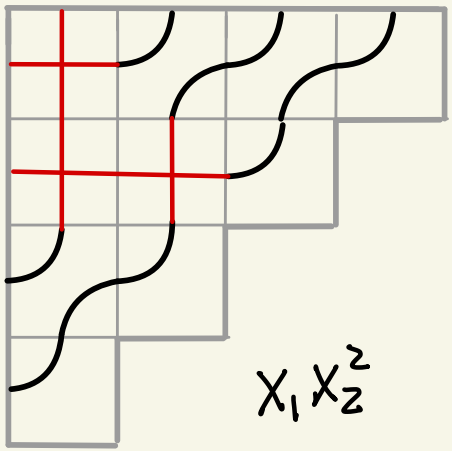
$X_1^2 X_2^2$



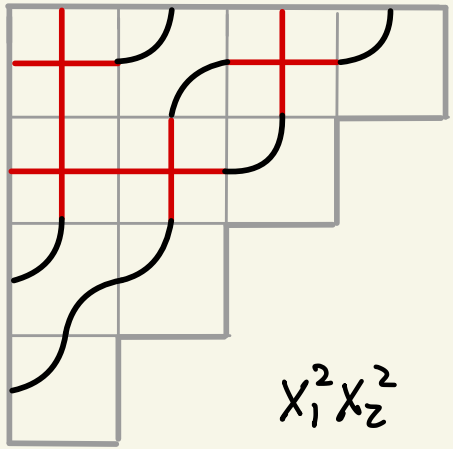
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Pipe Dreams

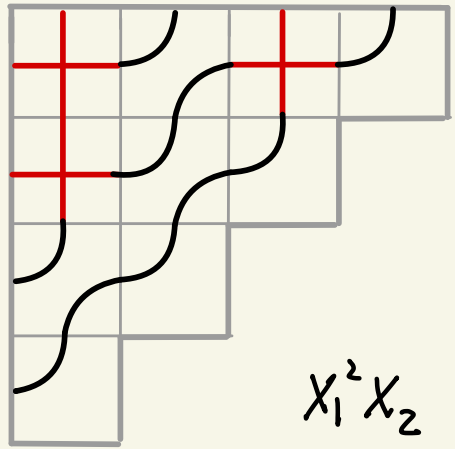
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$X_1X_2^2$



$X_1^2X_2^2$



$X_1^2X_2$

$$G_{2413} = X_1^2X_2 + X_1X_2^2 - X_1^2X_2^2$$

Bumpless Vertical-less Pipe Dreams

Top Grothendieck

$\widehat{Gw}$ = highest degree terms of Gw

Top Grothendieck

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1. Degree of $\widehat{Gw}$ determines the Castelnuovo-Mumford regularity of matrix Schubert varieties.
2. The support of $\widehat{Gw}$ conjecturally governs the support of Gw.

Top Grothendieck

Theorem [PSW21]

$\forall w \in S_n$. $\exists!$ inverse fireworks permutation u such that

$$\widehat{G_w} = \lambda_w \cdot \widehat{G_u}$$

Top Grothendieck

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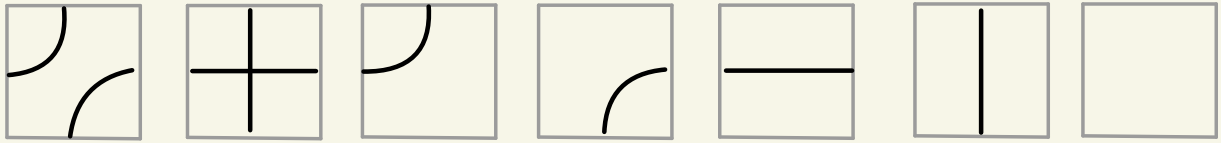
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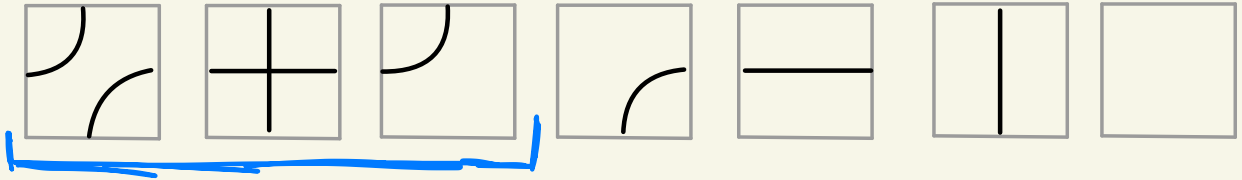
$$\lambda_w = \widehat{G_v}(1, 1, \cdots, 1)$$

v inverse fireworks

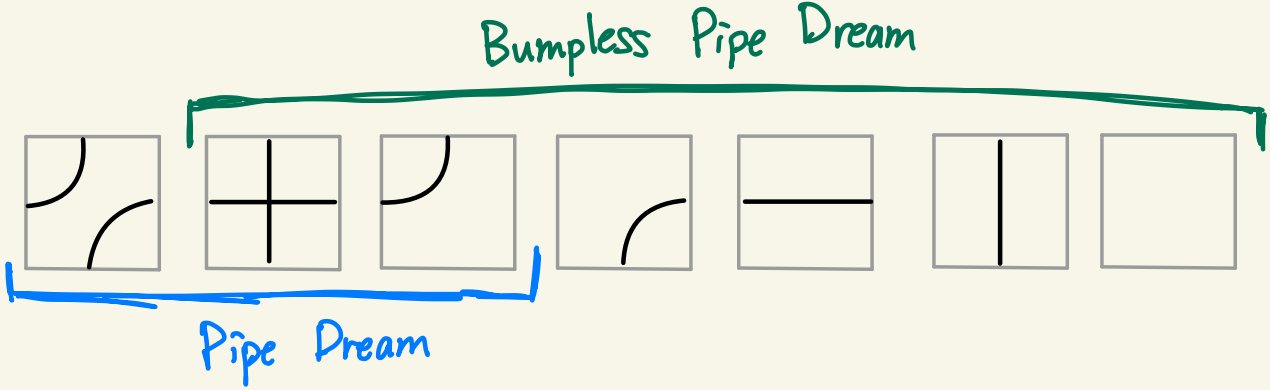
BVPD



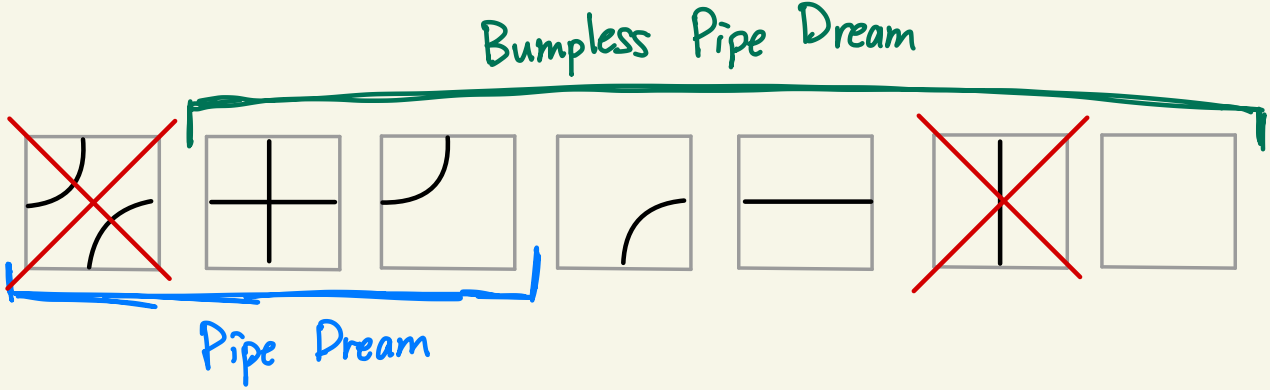
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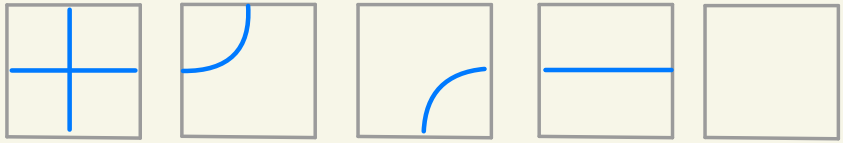


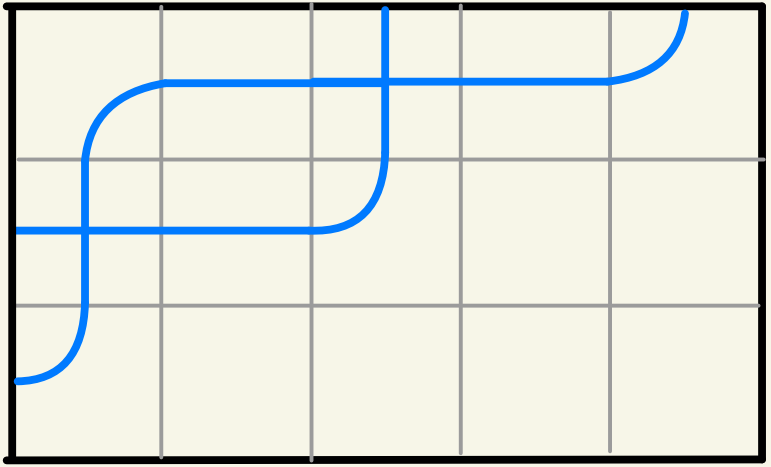
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Bumpless Vertical-less Pipe Dream

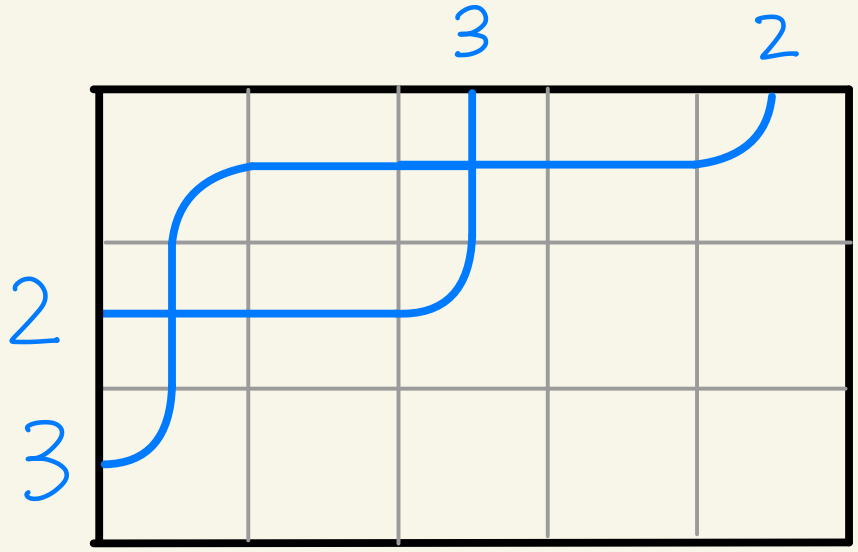
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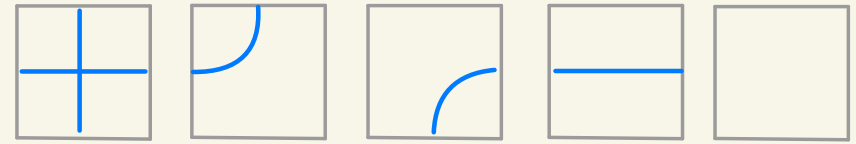


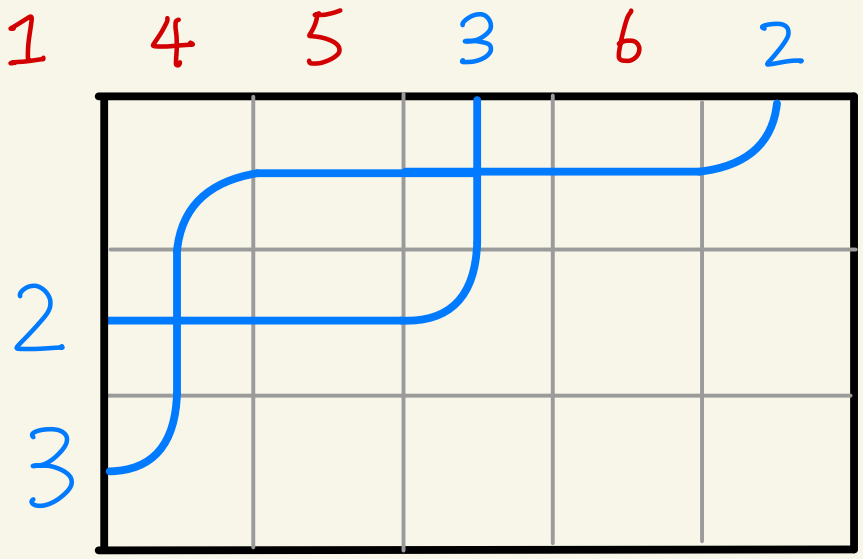
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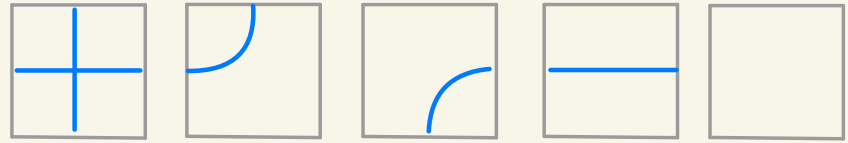


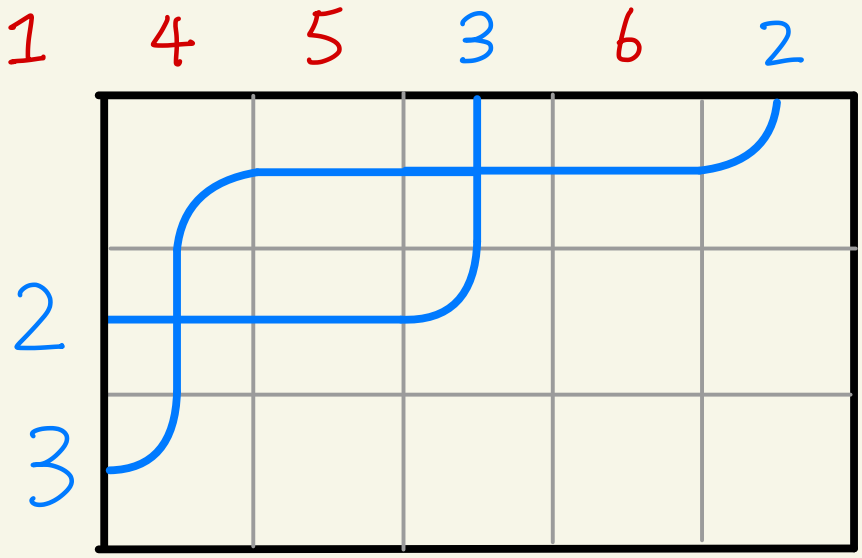
BVPD





BVPD





$w = 164235$

BVPD

Theorem [CY24]

For w inverse fireworks permutations,

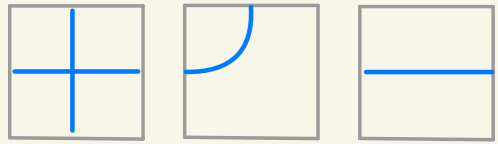
$$\hat{G}_w = \sum_{D \in \mathrm{BVPD}(w)} X^{\mathrm{wt}(D)}$$

BVPD

Theorem [CY24]:

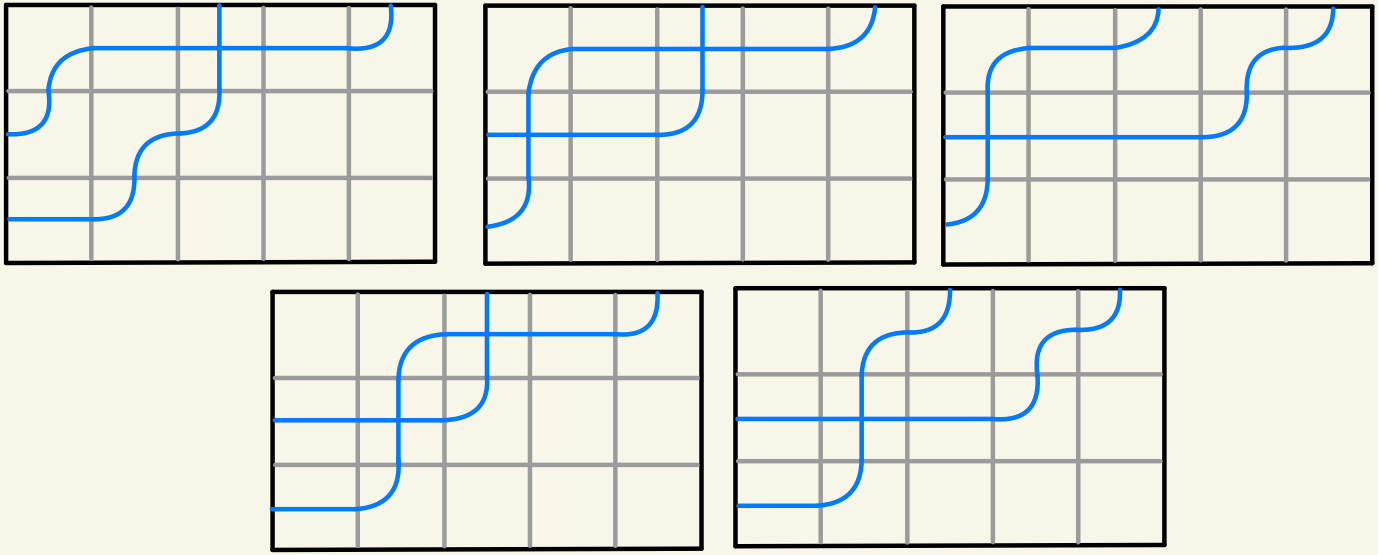
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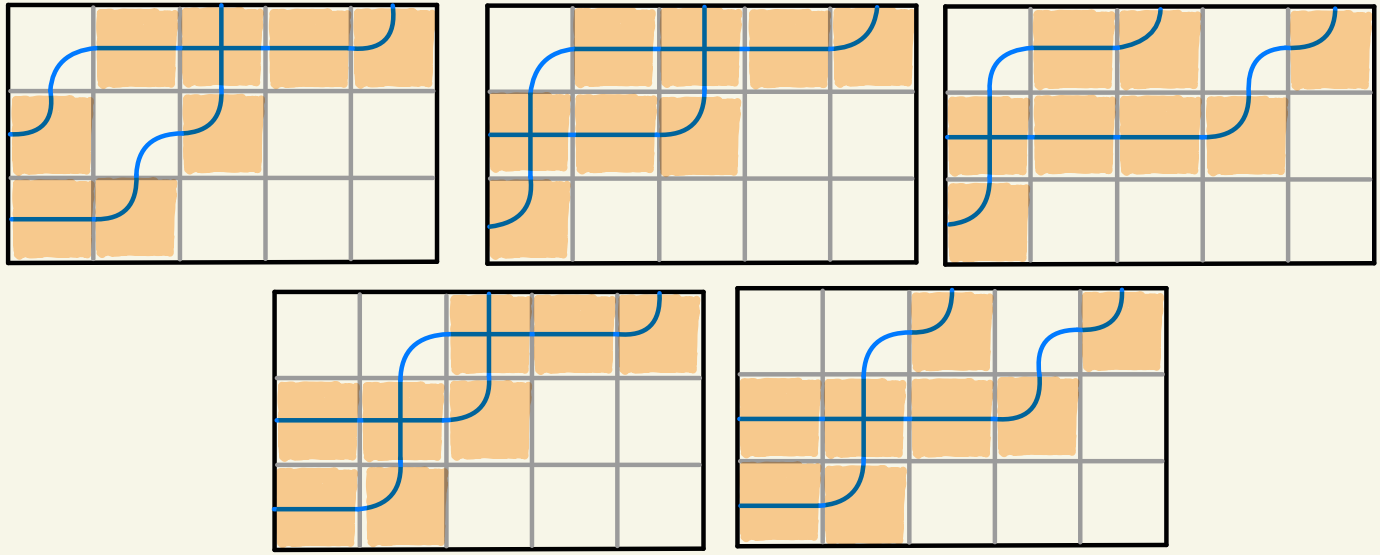
$\mathrm{wt}(D)$ = # of  in each row

BVPD

$w = 164235$ $w^{-1} = 145362$

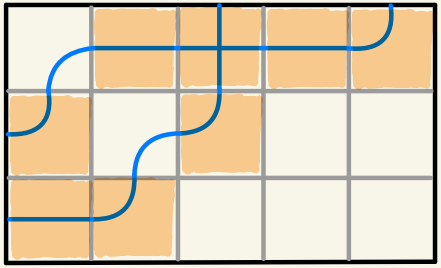


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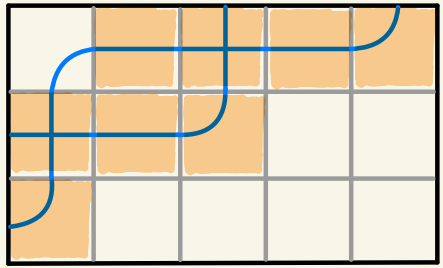


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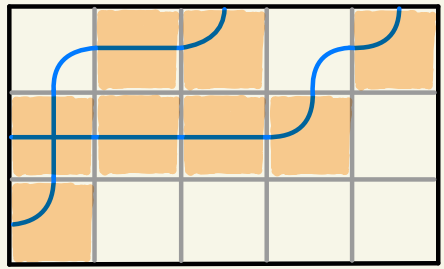
$x_1^4 x_2^2 x_3^2$



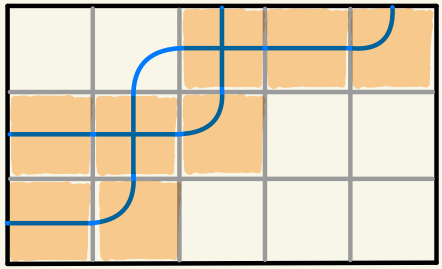
$x_1^4 x_2^3 x_3^1$

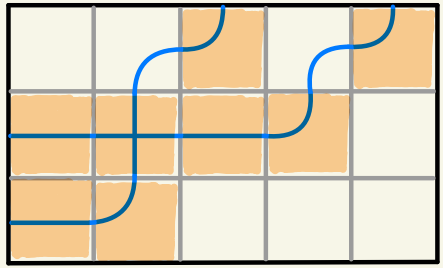


$x_1^3 x_2^4 x_3^1$



$x_1^3 x_2^3 x_3^2$

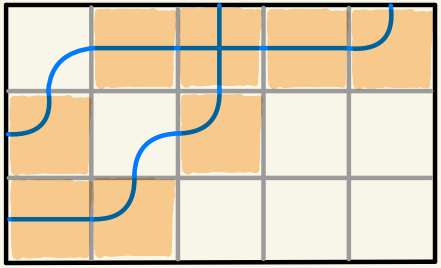




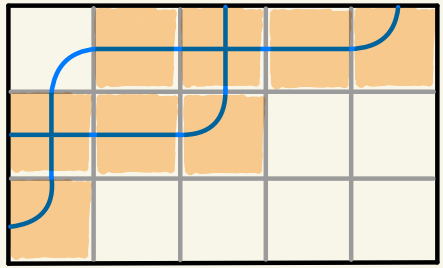
$x_1^2 x_2^4 x_3^2$

BVPD

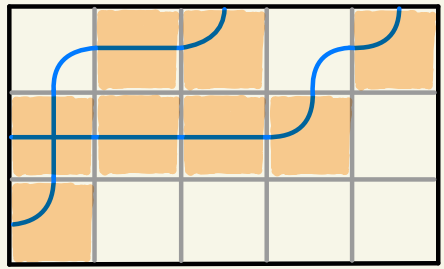
$x_1^4 x_2^2 x_3^2$



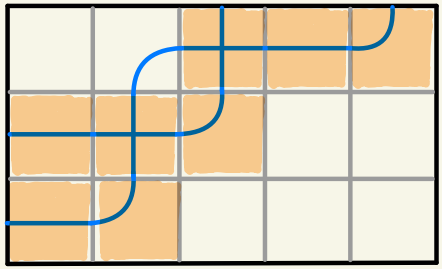
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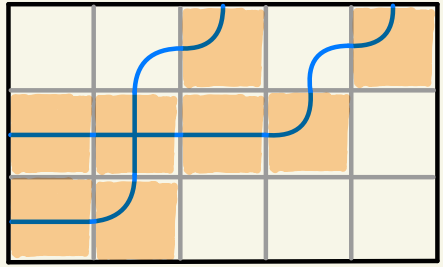


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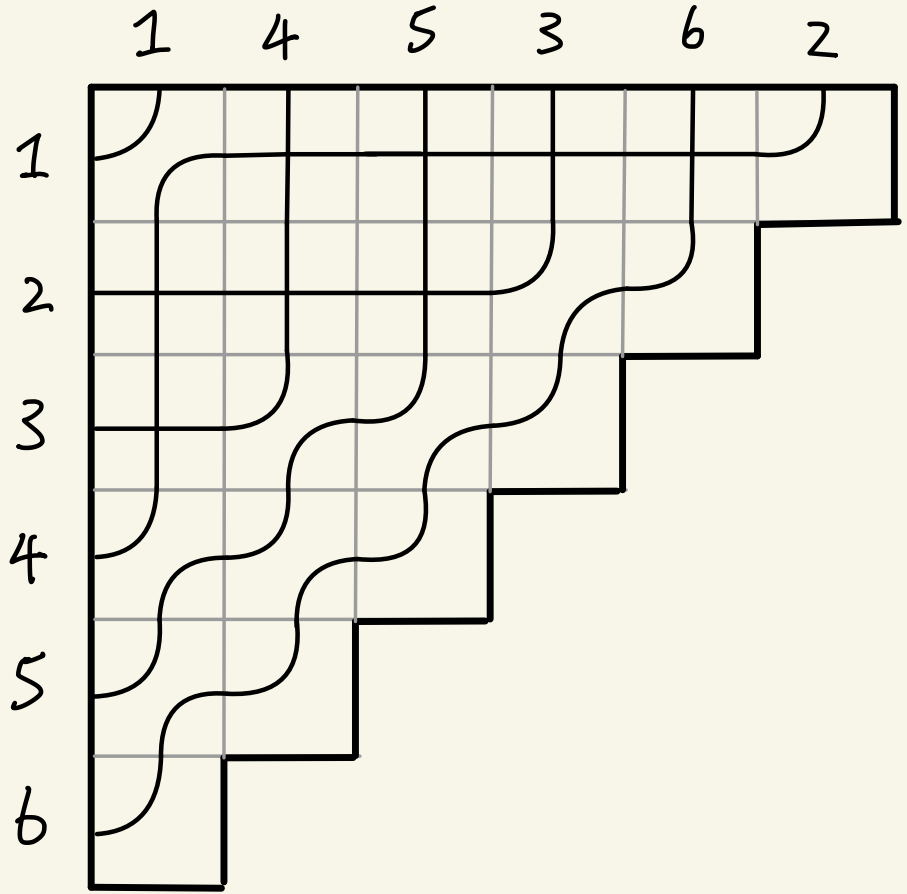




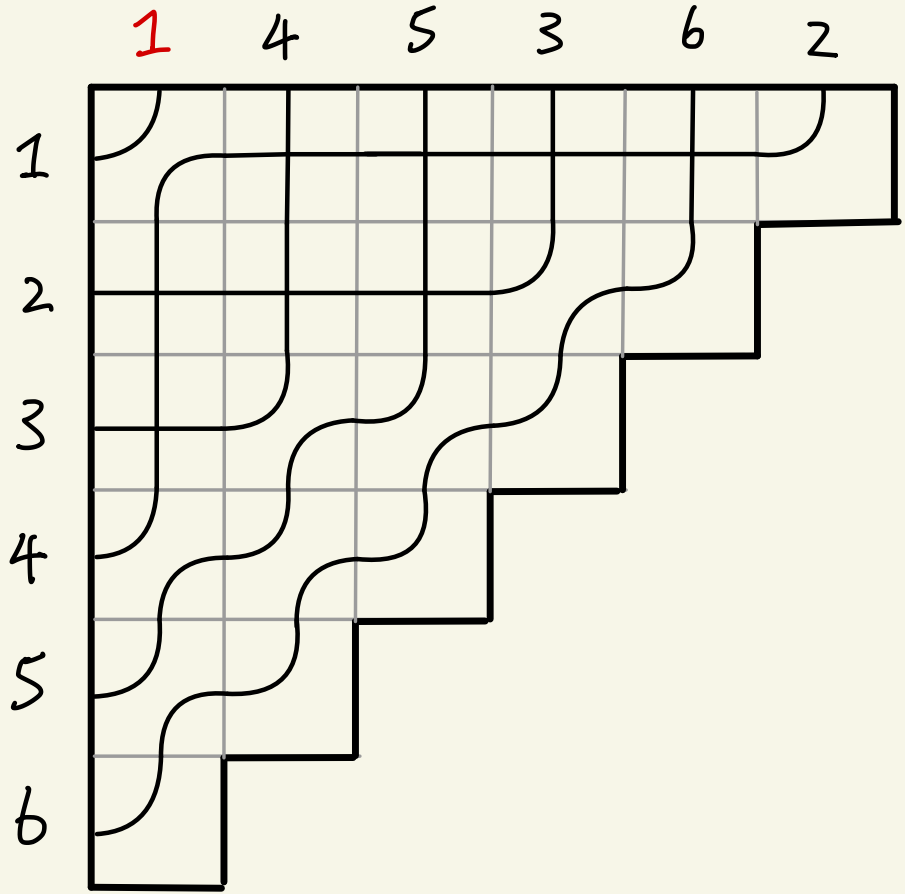
$x_1^2 x_2^4 x_3^2$

$$\widehat{G_W}(X) = x_1^4 x_2^2 x_3^2 + x_1^4 x_2^3 x_3^1 + x_1^3 x_2^4 x_3^1 + x_1^3 x_2^3 x_3^2 + x_1^2 x_2^4 x_3^2$$

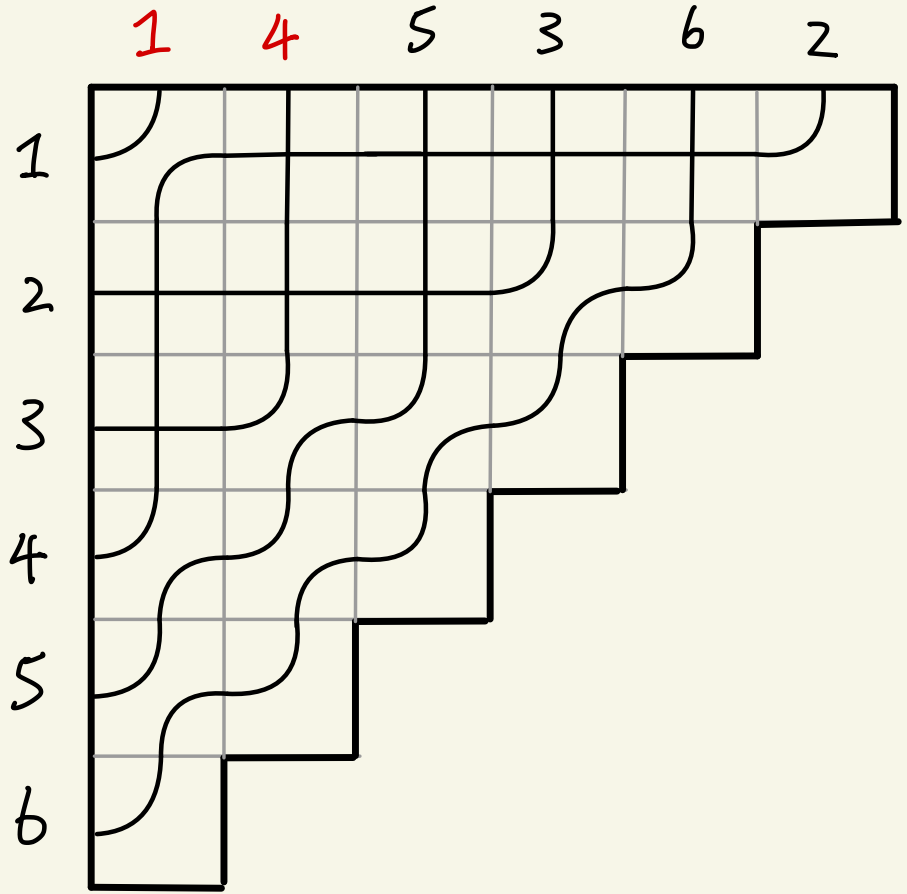
Sketch Proof



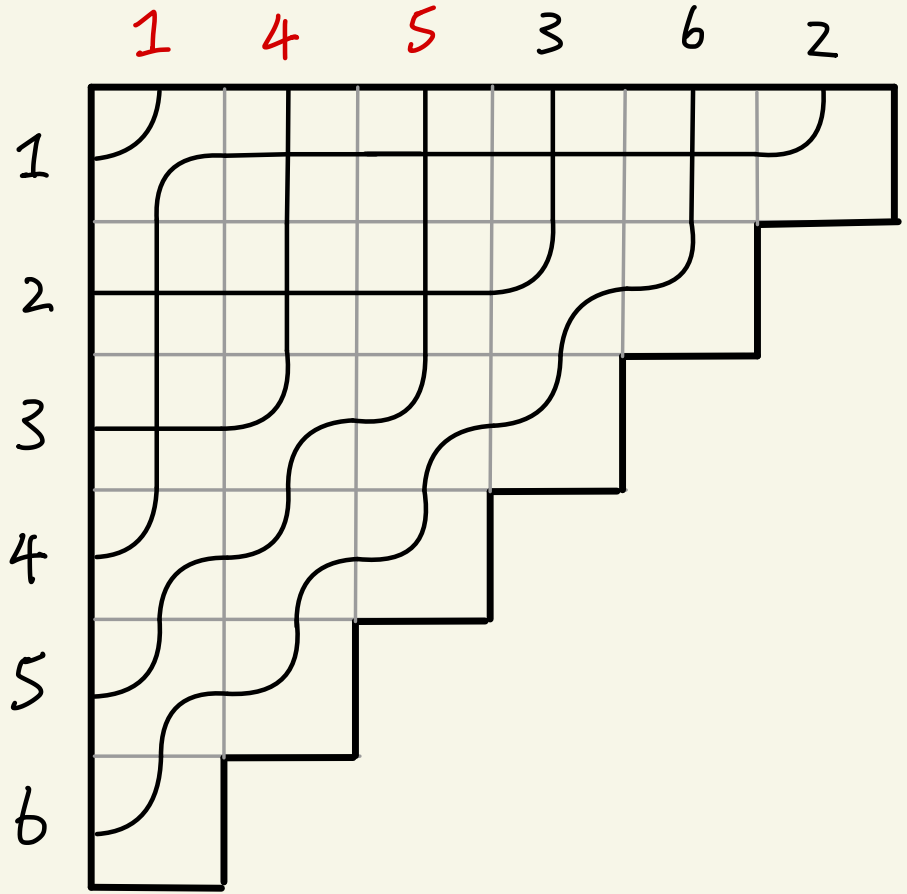
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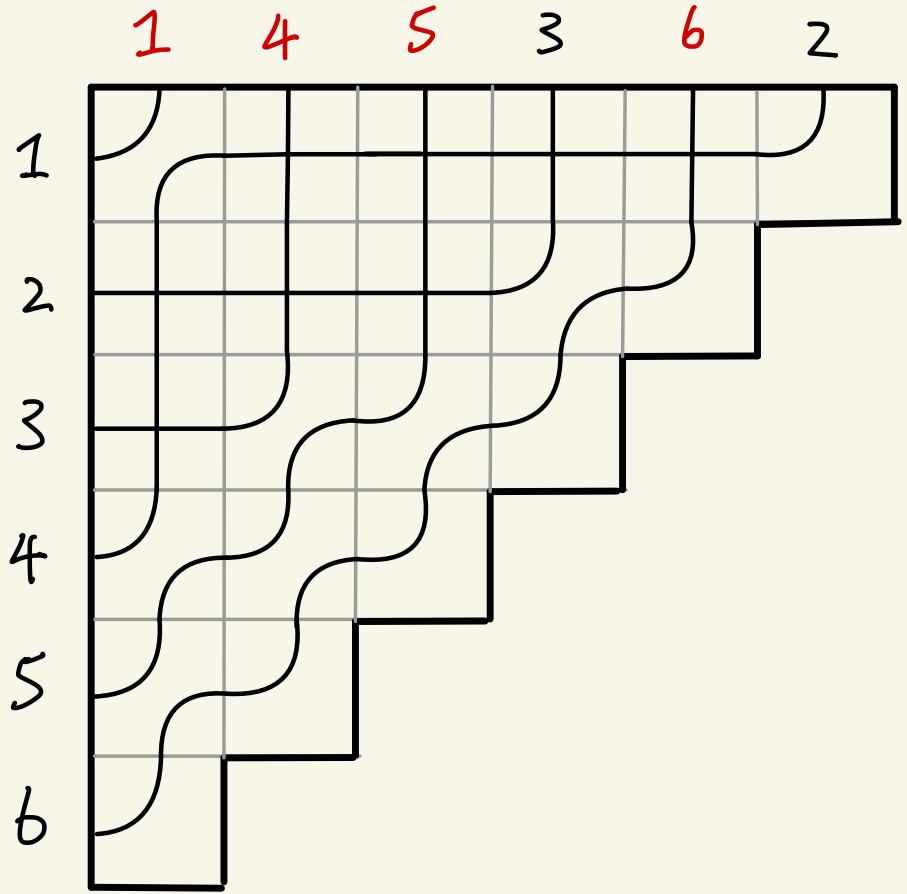
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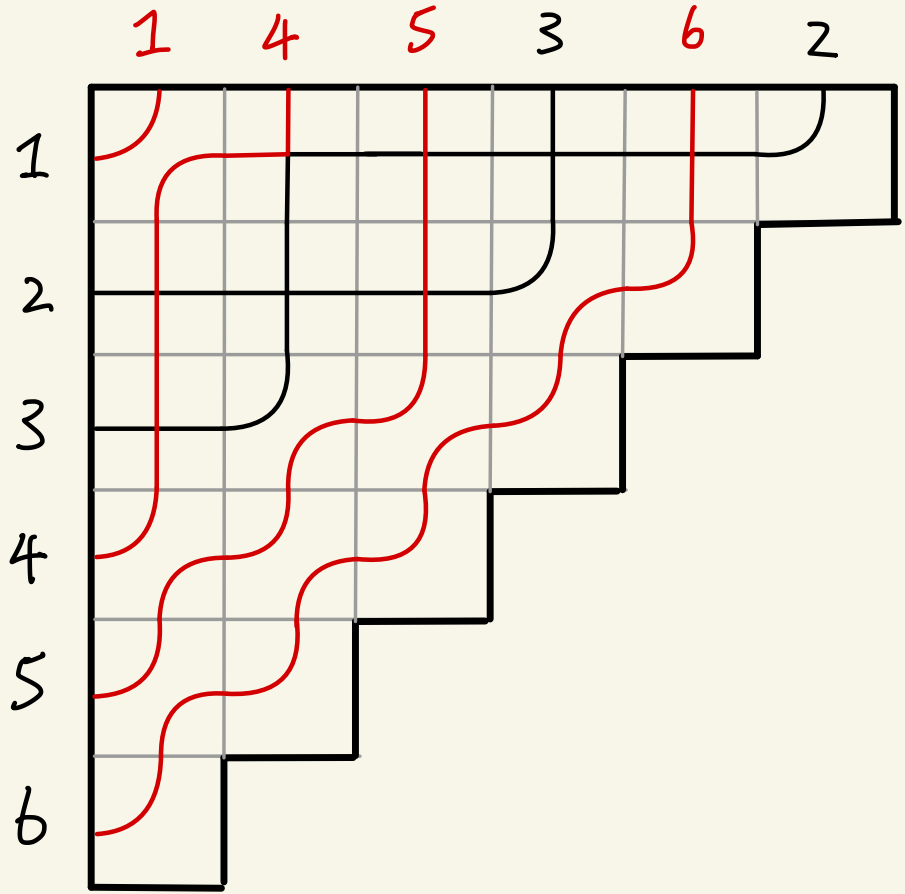
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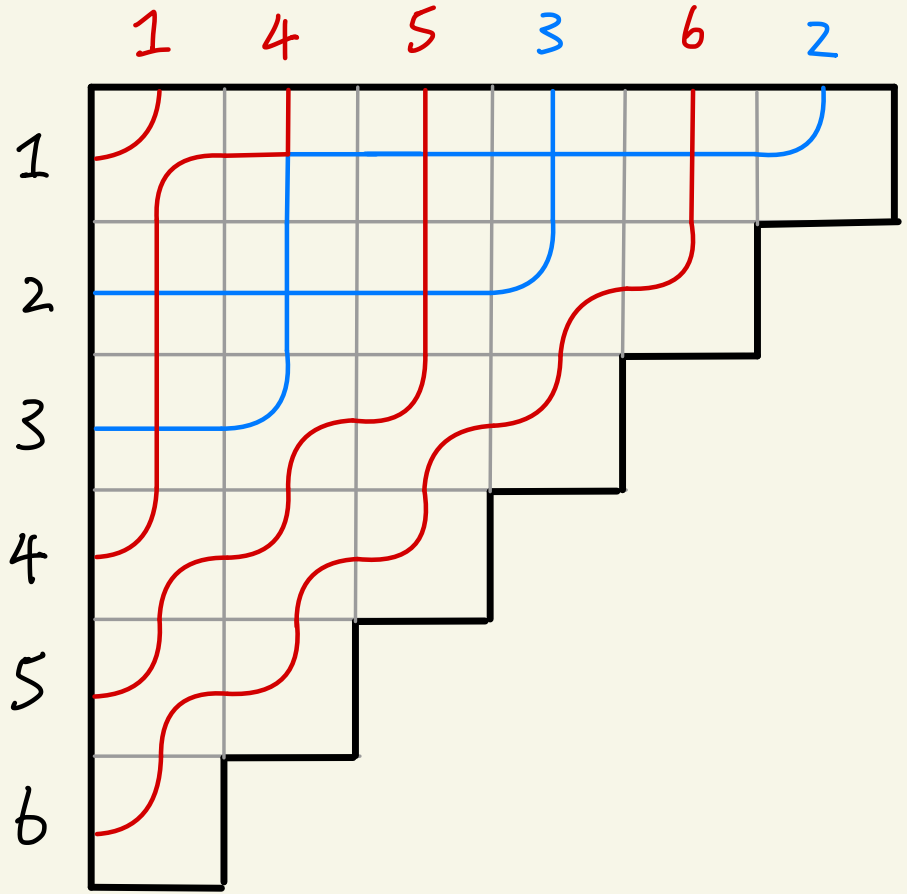
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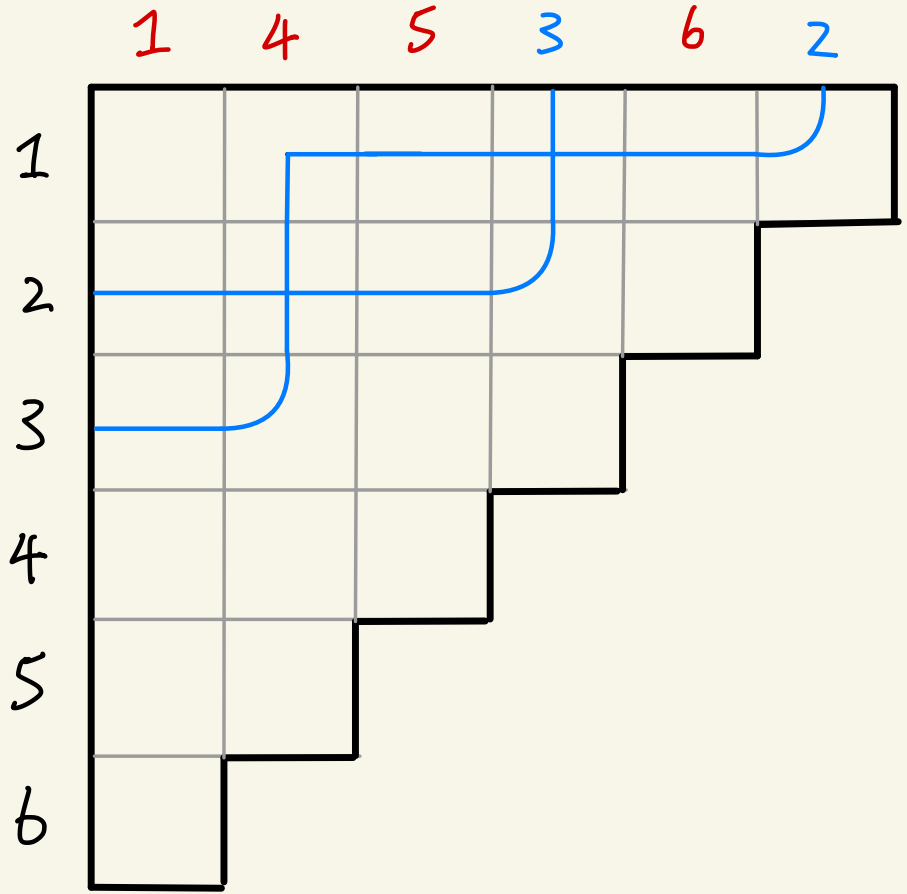
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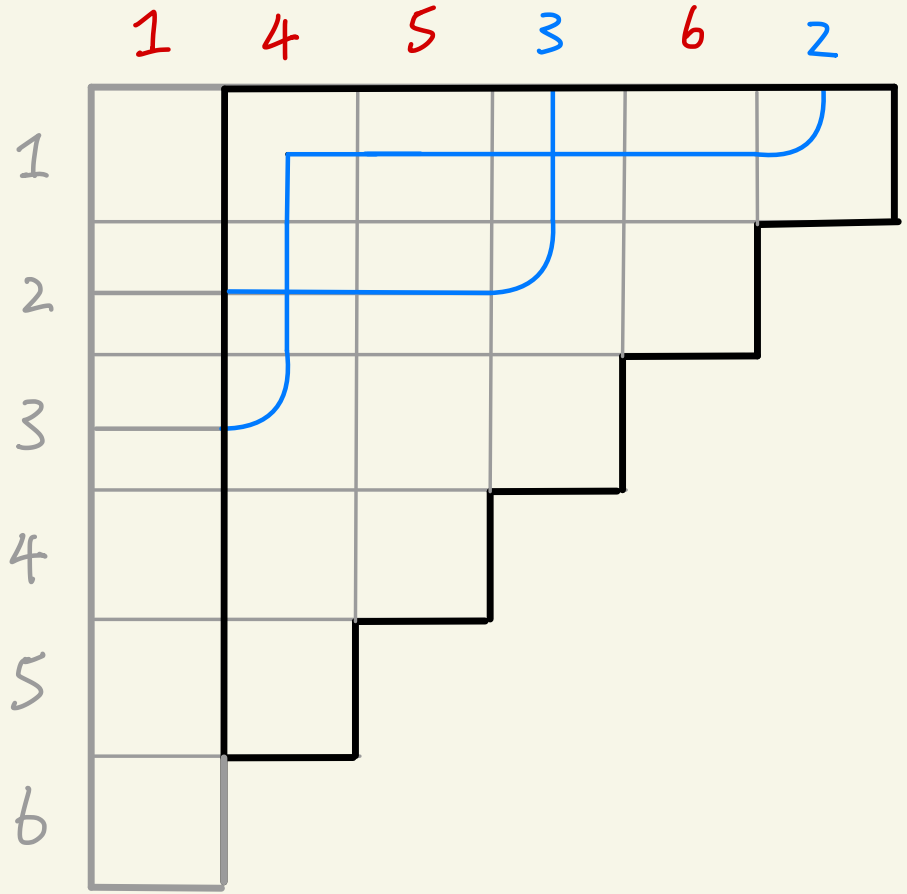
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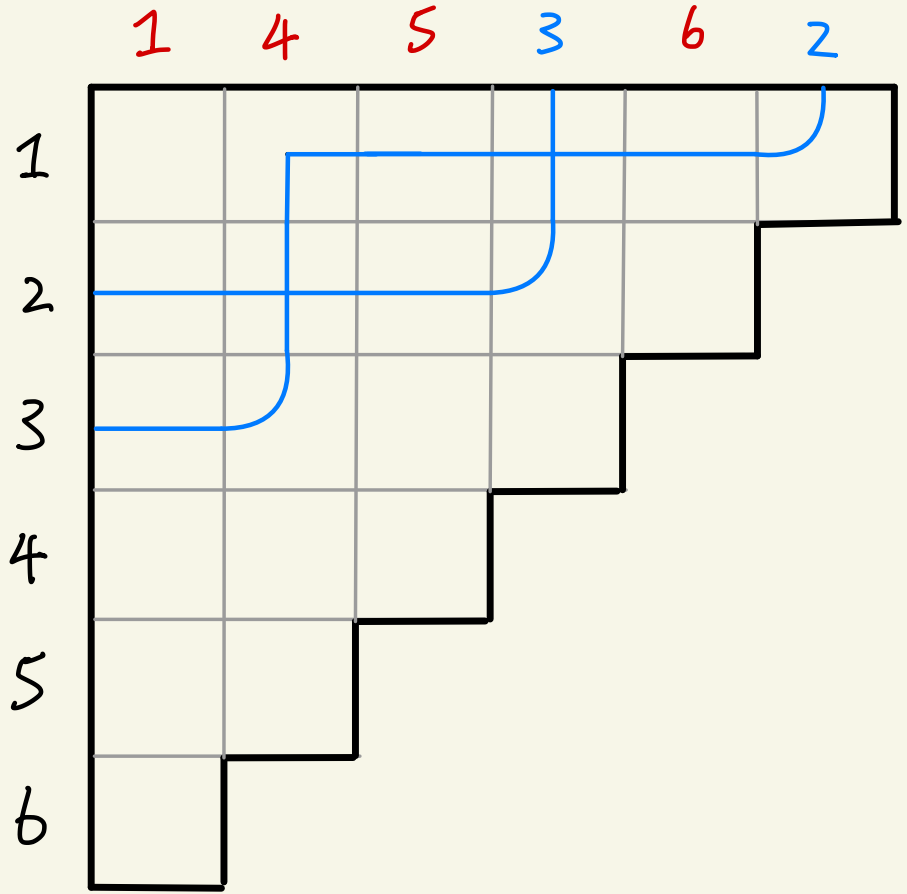
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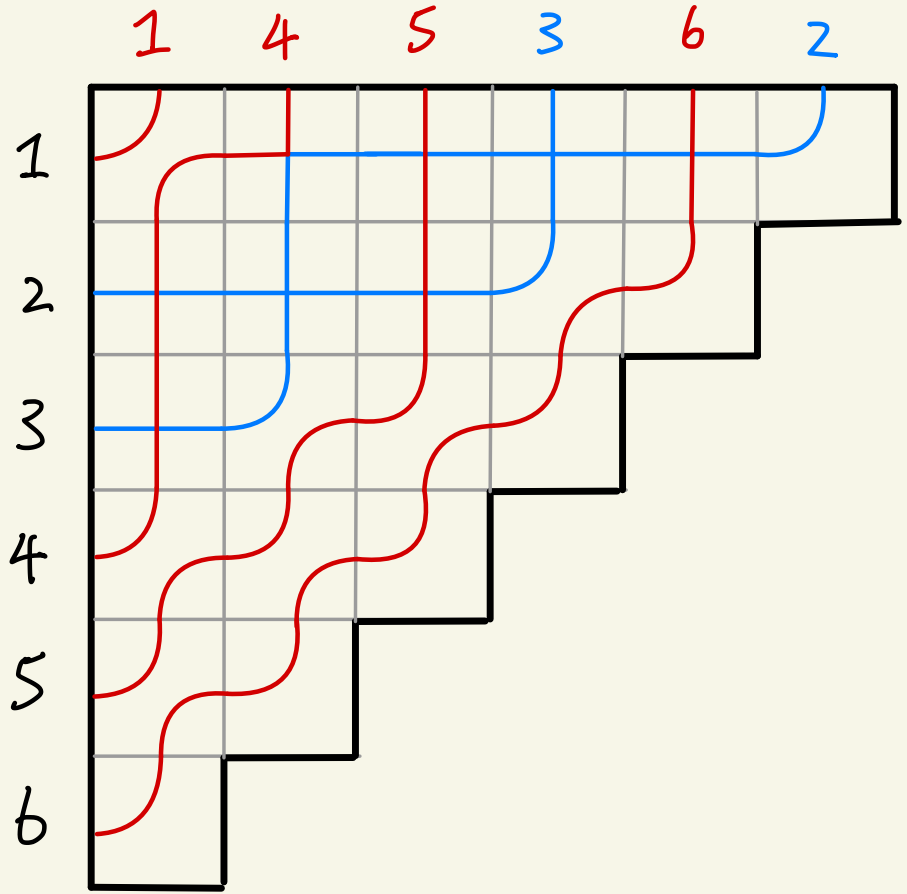
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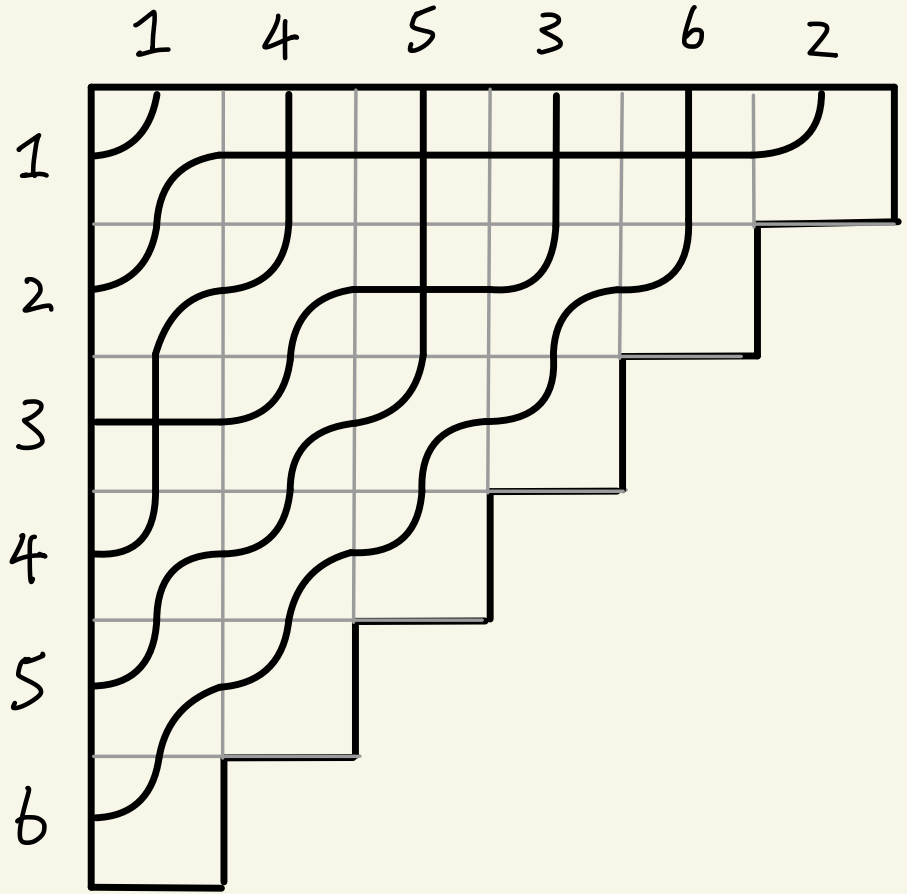
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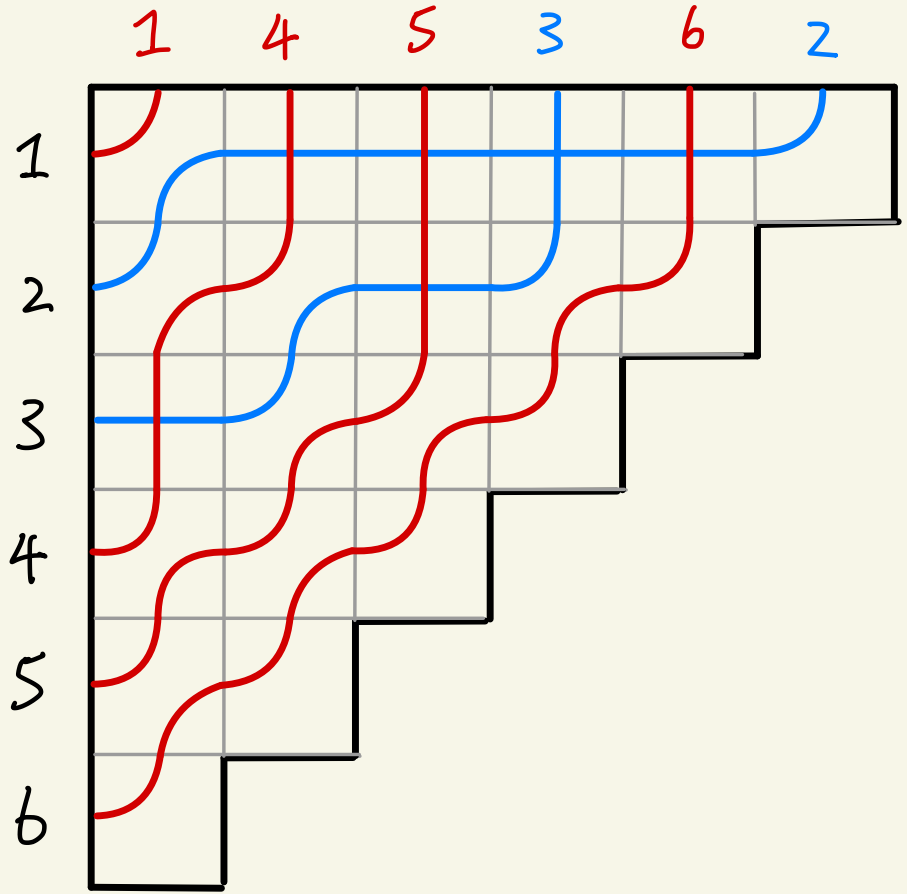
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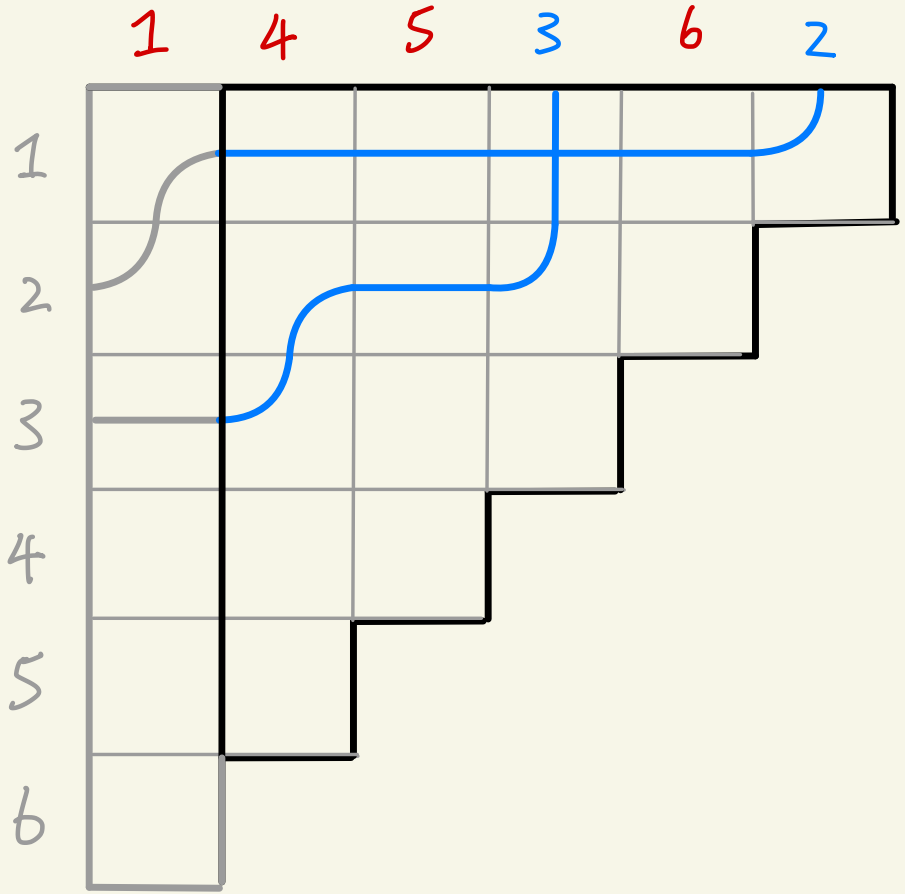
Sketch Proof



Sketch Proof

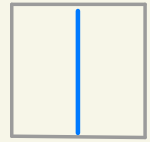


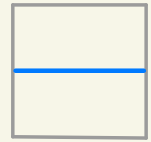
Sketch Proof



Sketch Proof

PD is heighest weight if and only if

1) Blue pipes have no  or 

2) Blue pipes in first column are 

Going-Up for Inverse Fireworks

Support of G_w

$$\text{Supp}(G_w) = \{\text{monomials in } G_w\}$$ ~~coefficients~~

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Big Important Conjecture [HMMSD 22]:

Homogenized Grothendieck polynomials

are Lorentzian.

Support of G_w

$$\text{Supp}(G_w) = \{\text{monomials in } G_w\}$$ ~~coefficients~~

Big Important Conjecture [HMMSD 22]:

~~Homogenized~~ Grothendieck polynomials

are ~~Lorentzian~~. "nice"

Support of G_w Conjectures [MSSD24]

1. If $m \in \mathrm{Supp}(G_w)$ not highest degree, then $\exists\, m' \in \mathrm{Supp}(G_w)$ of higher degree s.t. $m \mid m'$.

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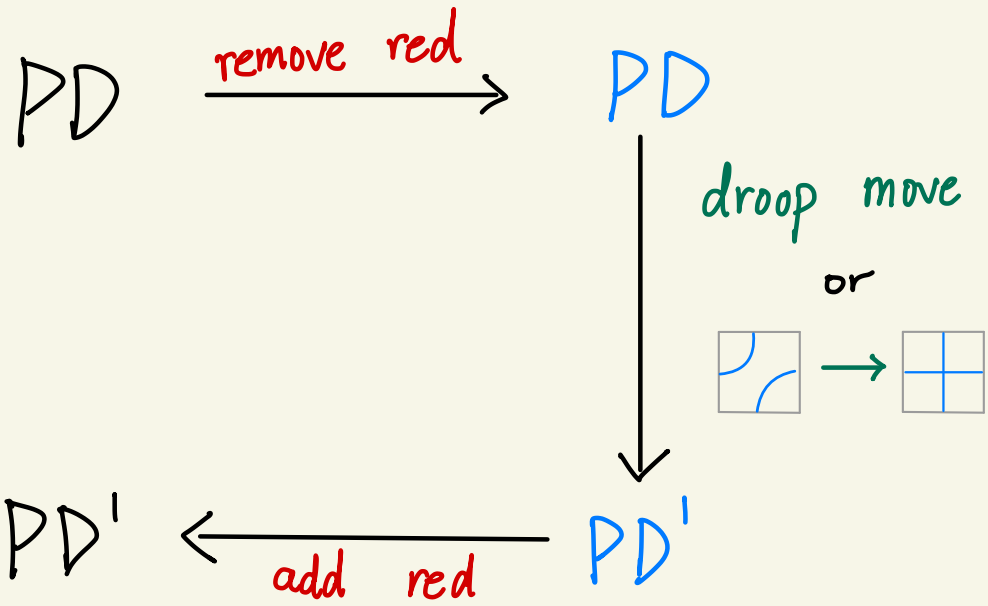
[Fireworks, vexillary, Zero-One, Inverse fireworks]

Going Up

Question:

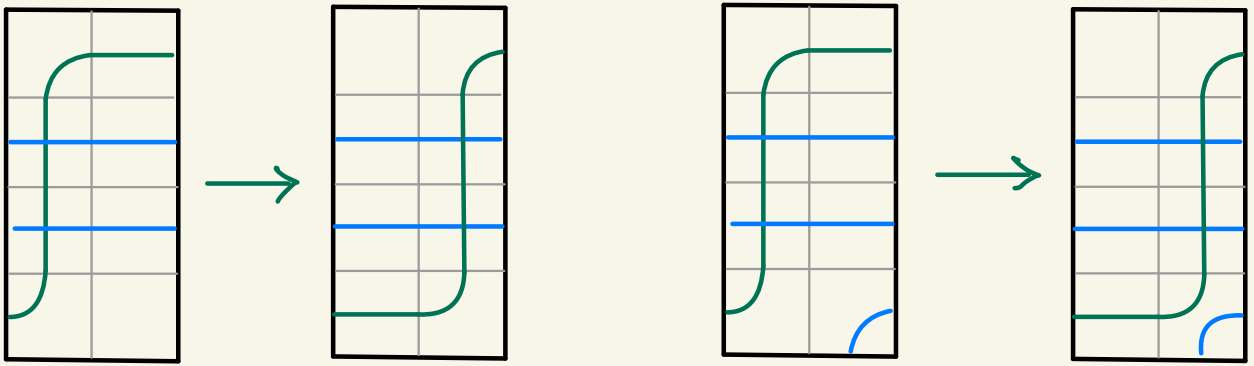
Given a PD not of highest weight, how can we increase the weight of one of the rows by exactly one?

Going Up



Going Up

Droop moves



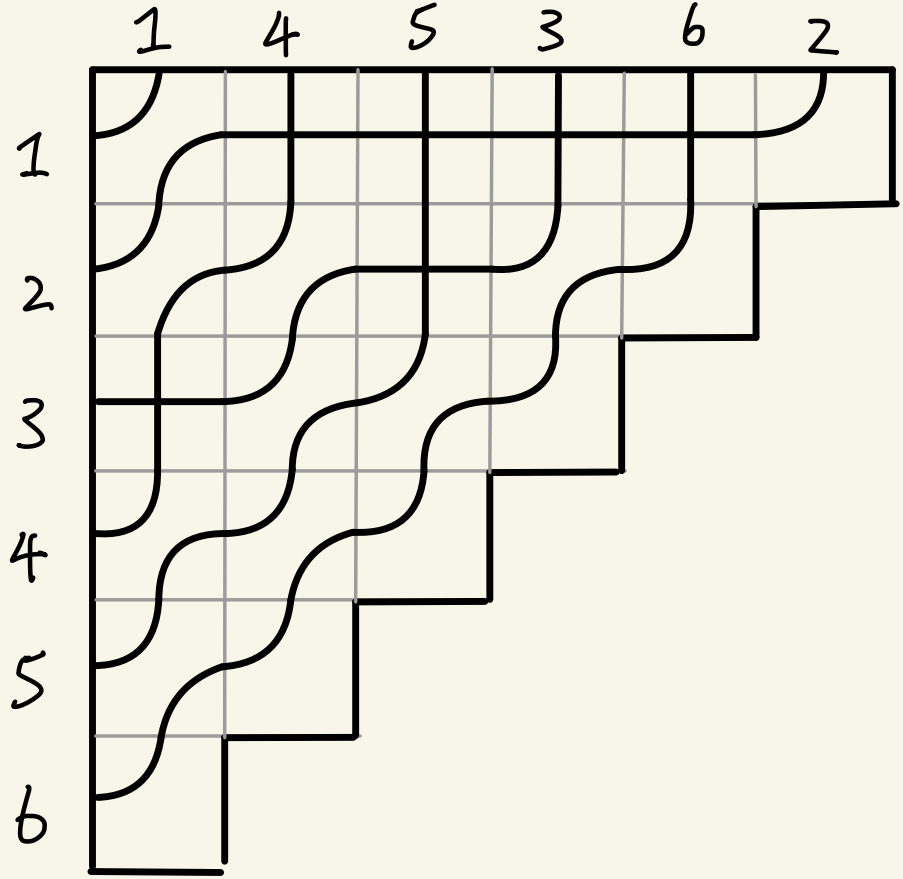
Going Up

Lemma:

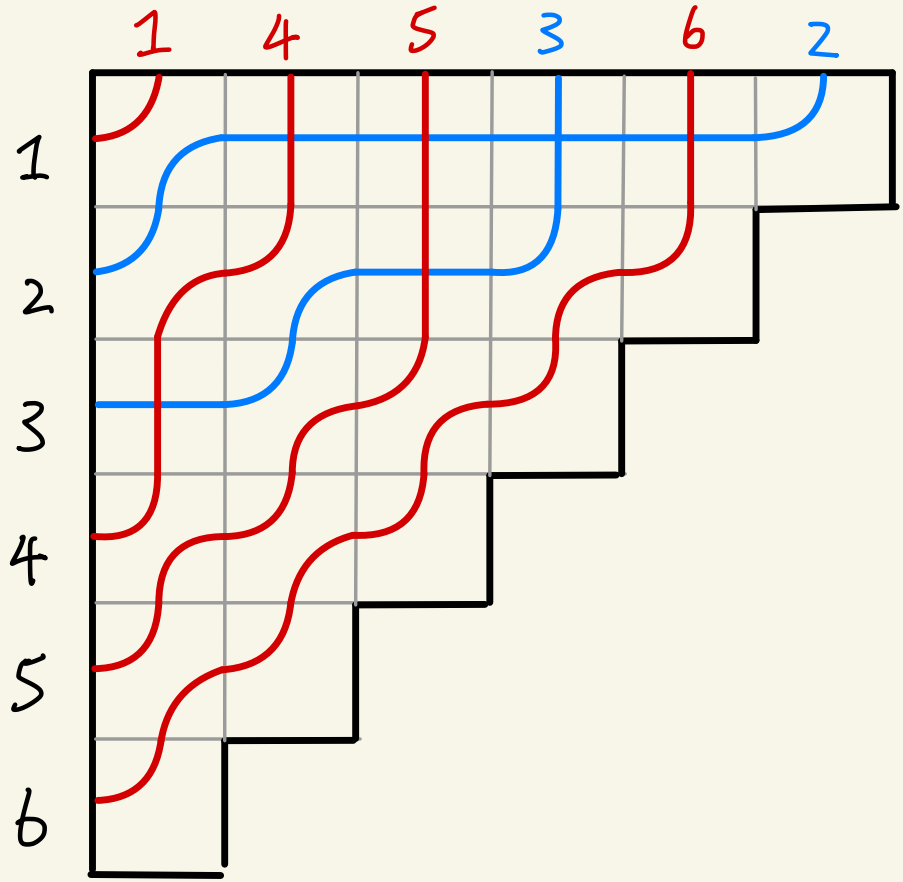
If PD is not highest weight, then we must be able to change  or do a droop move.

After a droop move, either the weight doesn't change or one of the rows increases by 1.

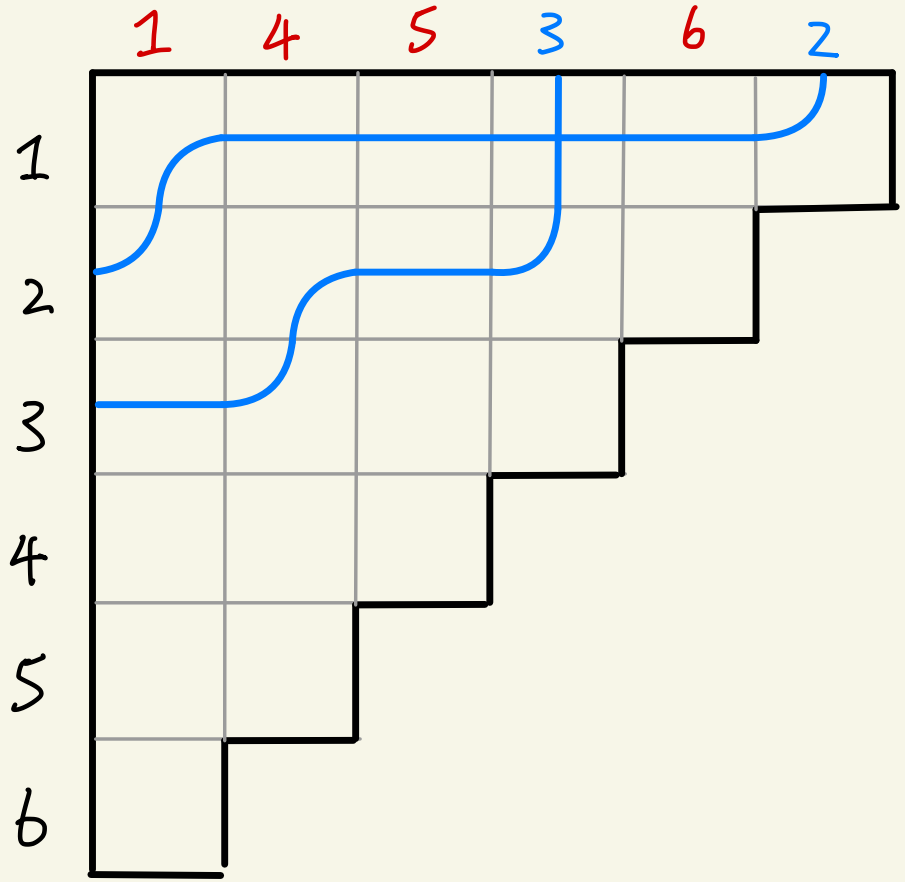
Going Up



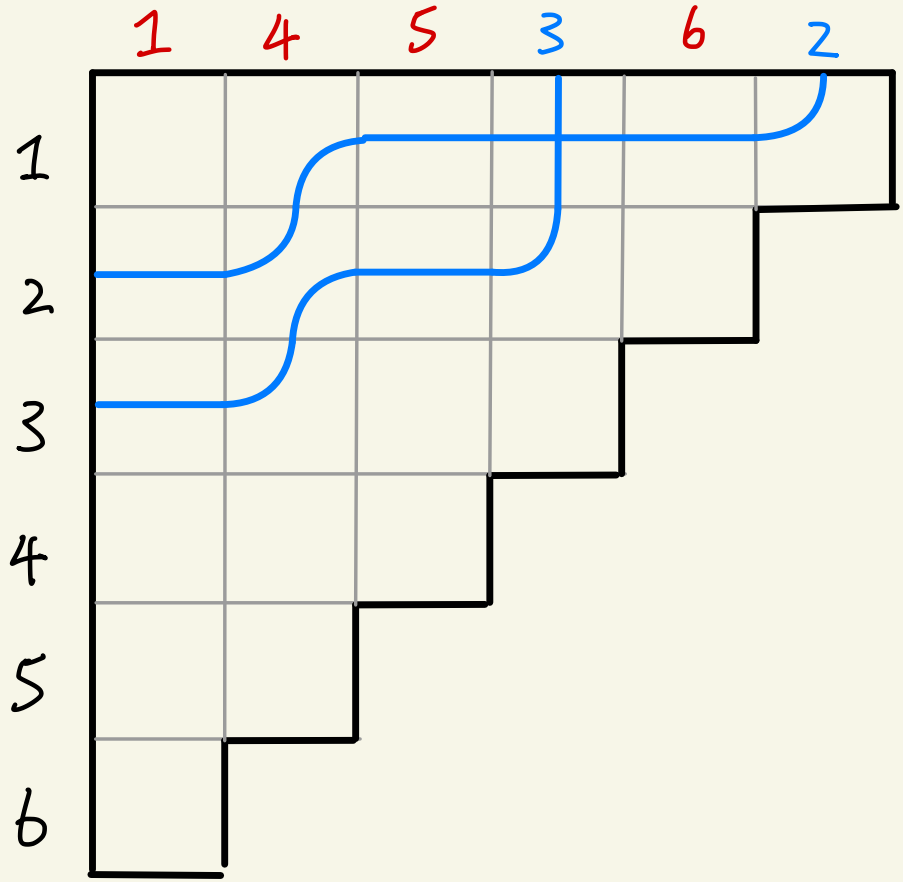
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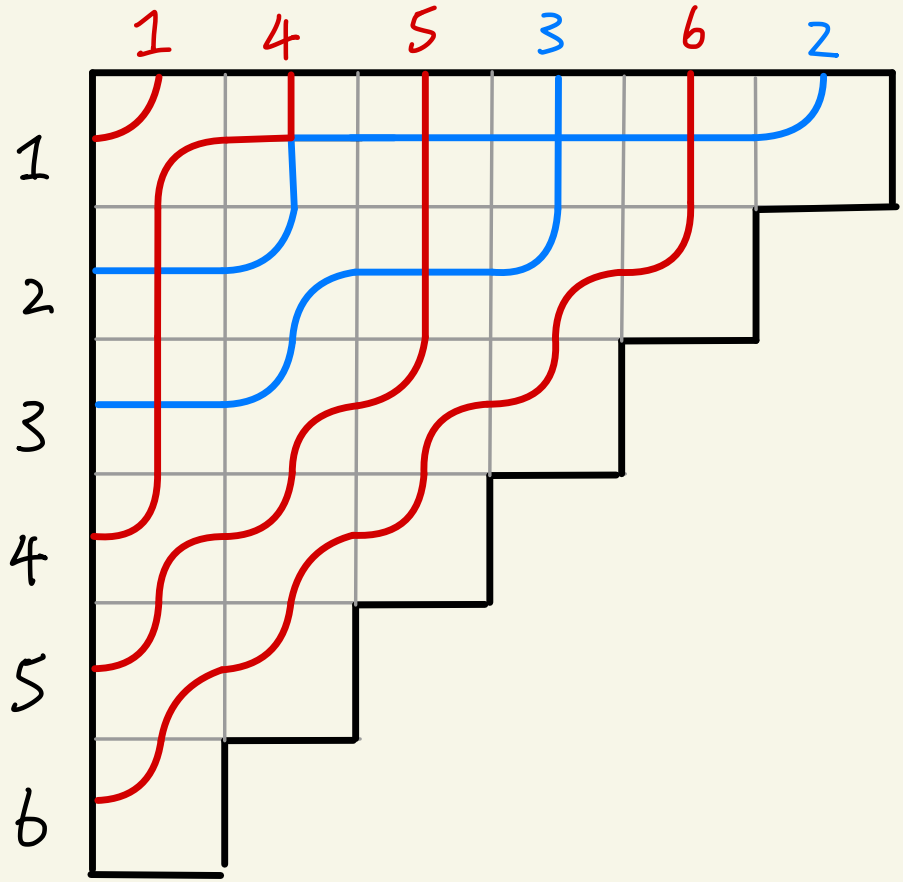
Going Up



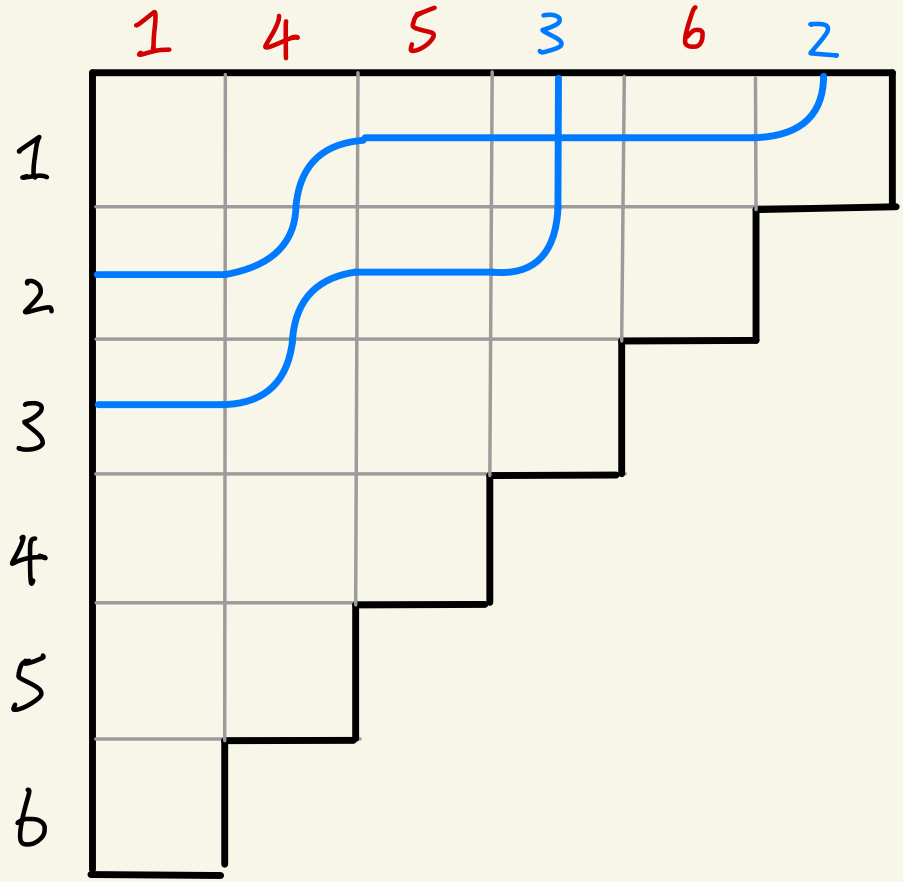
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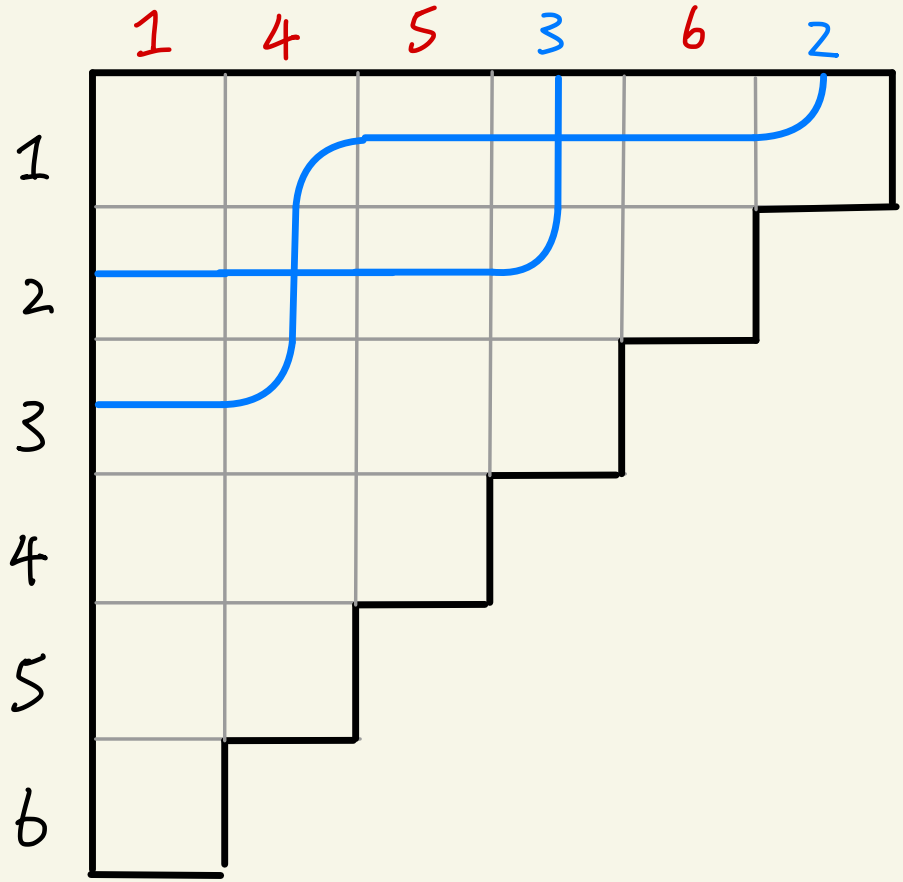
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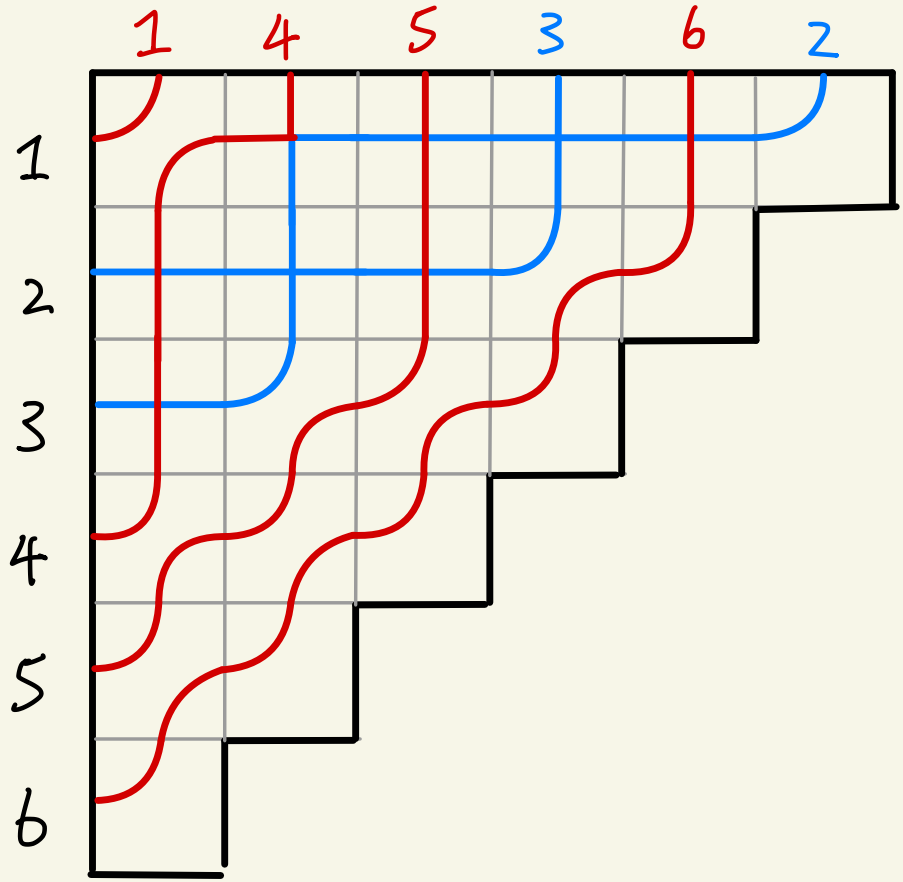
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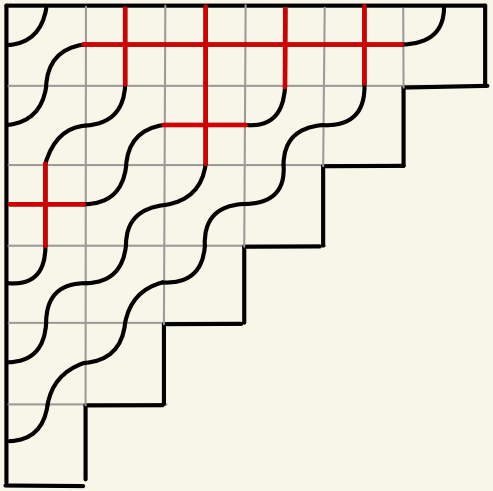
Going Up

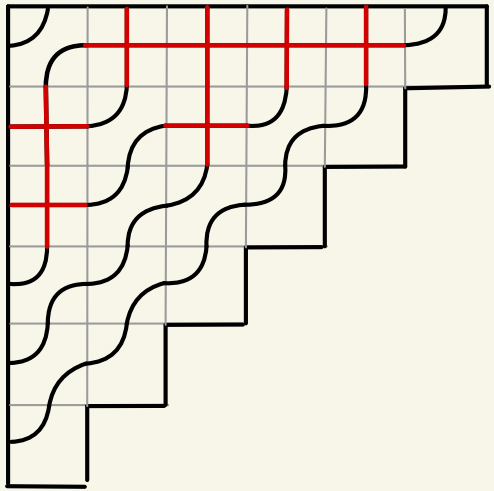


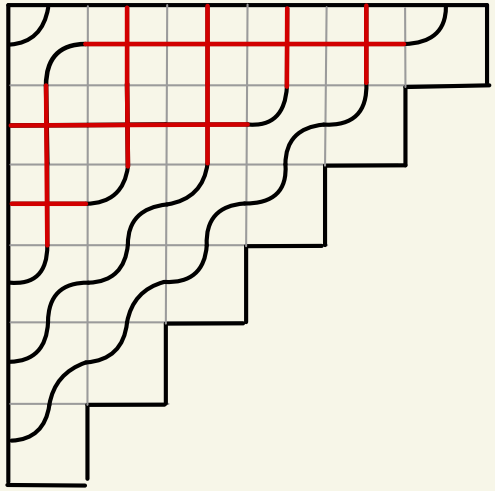
Going Up



Going Up







Going Up

If $m \in \text{Supp}(G_w)$ not highest degree, then $\exists\, x_i$ such that $x_i m \in \text{Supp}(G_w)$.

Going Up

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Thm [CY24]:

True for inverse fireworks permutations.



Double Grothendieck Polynomials

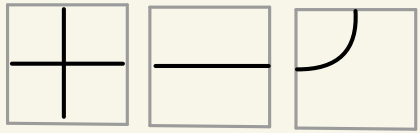
Double Grothendieck $G_w(x;y)$

$$G_w(x;y) = \sum_{D \in PD(w)} \prod_{\boxplus \in D} (x_i + y_j - x_i y_j)$$

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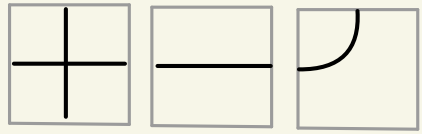
$$\hat{G}_w(x;y) = \sum_{D \in BVPD(w)} \prod_{(i,j) = wt} (x_i y_j)$$



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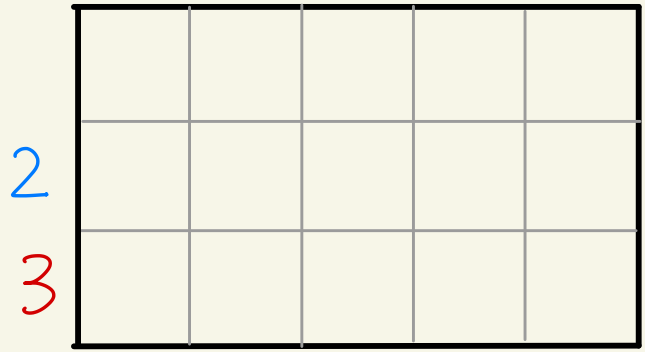
$$\hat{G}_w(x;y) = \sum_{D \in BVPD(w)} \prod_{(i,j) = wt} (x_i y_j)$$



$$G_w(x;0) = G_w(x) \qquad \hat{G}_w(x;1) = \hat{G}_w(x)$$

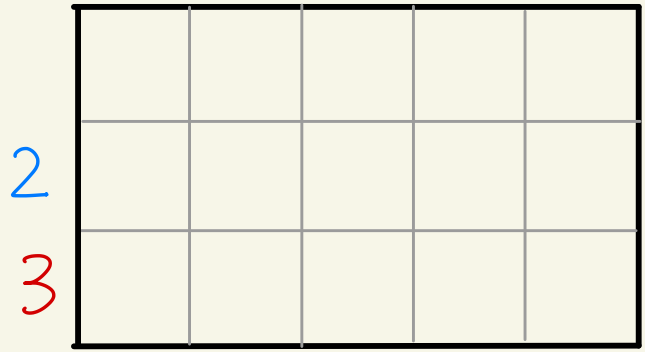
$w = 164235$

$w^{-1} = 1\ 4\ 5\ 3\ 6\ 2$



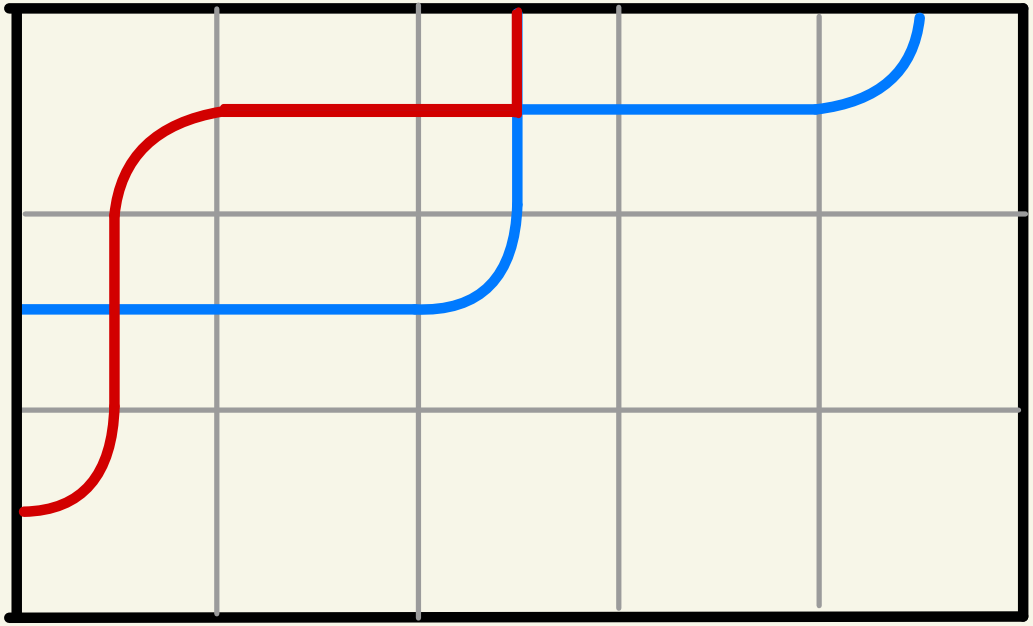
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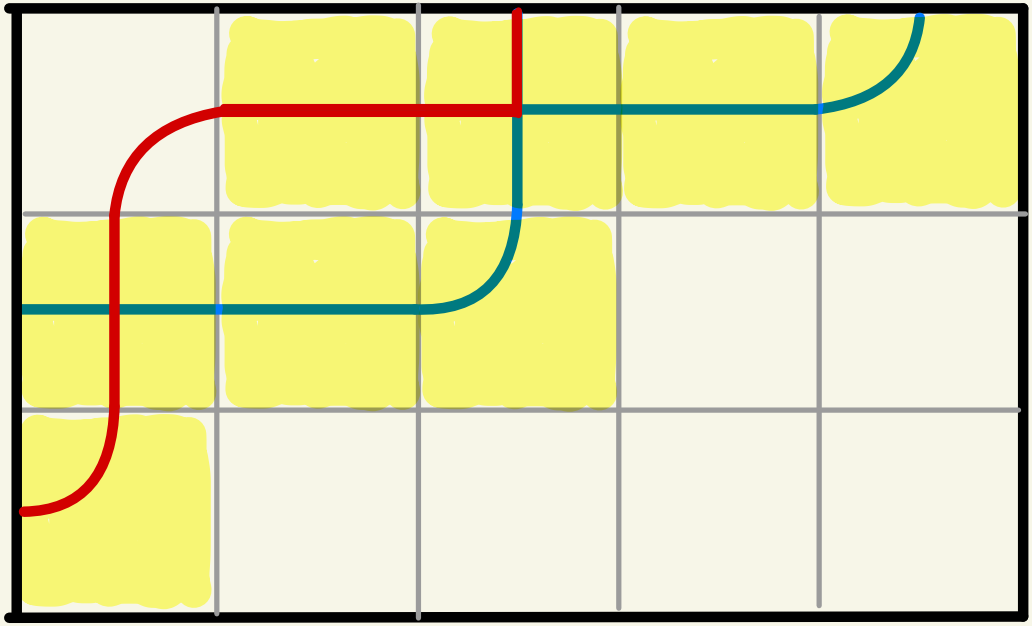
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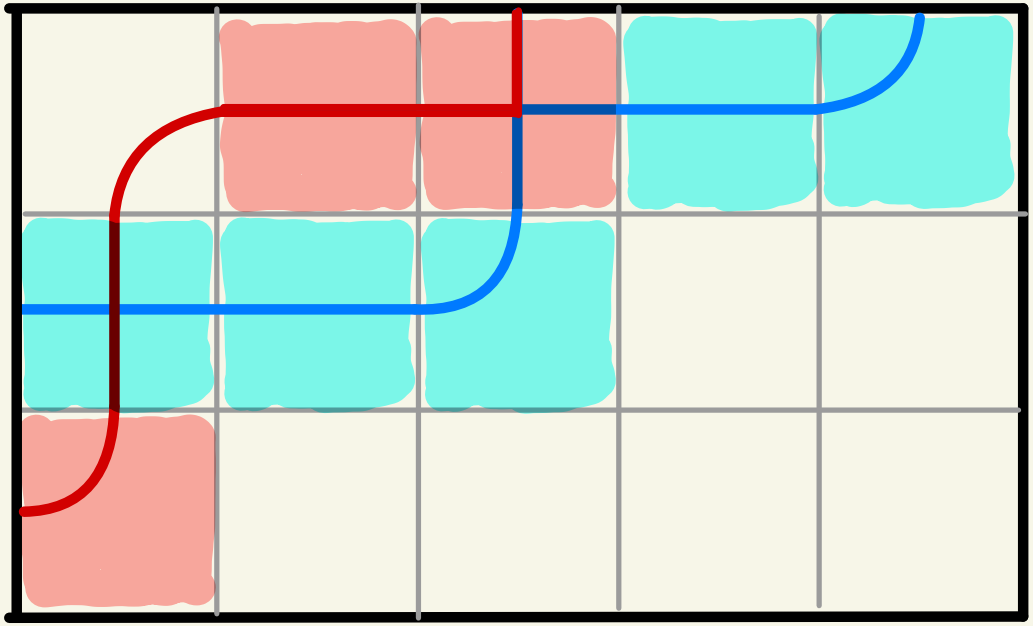


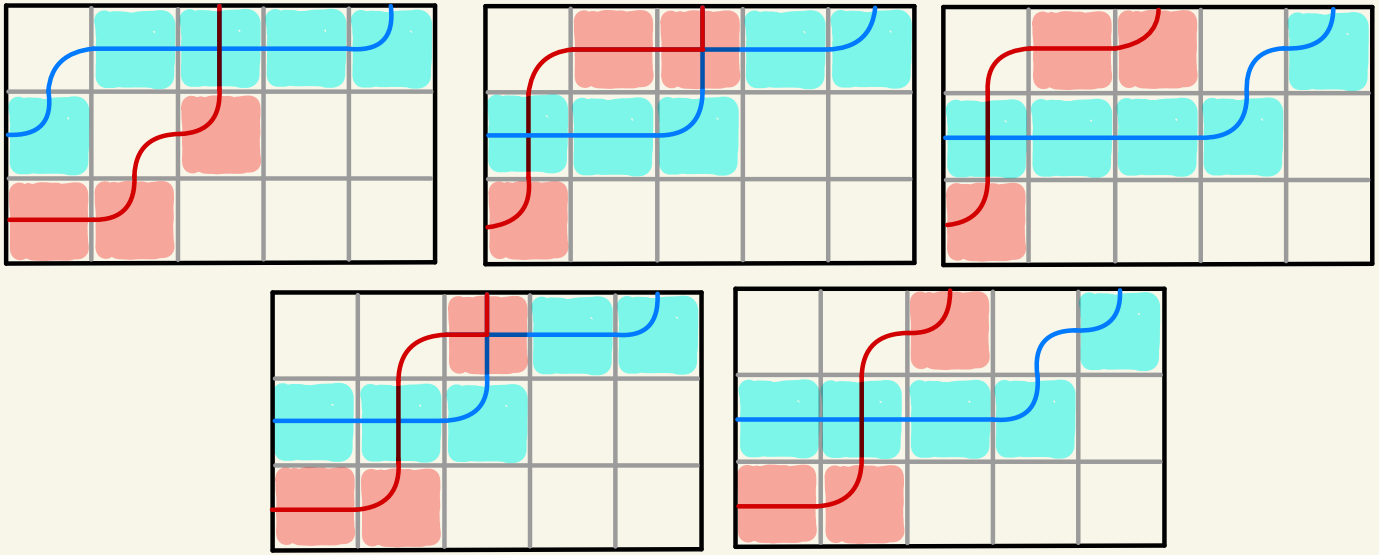
$$\widehat{G_w}(1;y) = \lambda_w \cdot y_1 y_2 y_3 y_1 y_2 y_3 y_4 y_5 = \lambda_w \cdot y_1^2 y_2^2 y_3^2 y_4 y_5$$

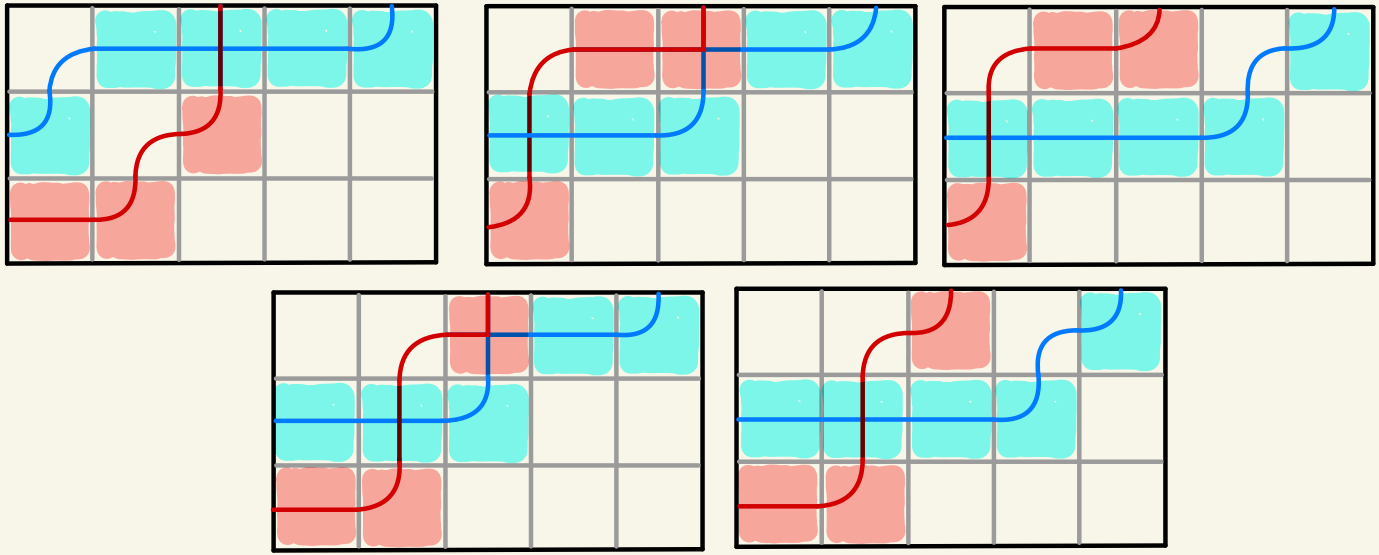
$$\deg(G_w(x)) = \deg(\widehat{G_w}(x)) = \deg(\widehat{G_w}(1;y)) = 3+5 = 8$$











$$\widehat{G_w}(1,y) = 5\, y_1 y_2 y_3\, y_1 y_2 y_3 y_4 y_5 = 5\, y_1^2 y_2^2 y_3^2 y_4 y_5$$

$$\deg(G_w(x)) = 3 + 5 = 8$$

w Invfire $\Longrightarrow$ $\widehat{G_w}(1,Y)$ is a single monomial

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$\Longleftrightarrow$ $\widehat{G_w}(X) = G_u(X)$ Dominant

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$\Longleftrightarrow$ $\widehat{G_w}(X) = G_u(X)$ Dominant

$\Longleftrightarrow$ Degree of leading monomial of $G_w(X)$ is weakly decreasing.

Thank You!



