Newton Polytopes of Fireworks Grothendieck Polynomials

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RIMS, Kyoto University
July 17th 2025

Outline

1. Fireworks permutations & Grothendieck polynomials
2. Support
3. Newton Polytope
4. Layered permutations

Fireworks

The start of each decreasing run is increasing.

3 1 6 7 5 4 2

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3 < 7 ≮ 6

Fireworks

Generalized pattern avoidance : 3_12 avoiding

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Bell numbers B(n) : 3167542 $\longleftrightarrow$ 13|2457|6

Fireworks

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Bell numbers B(n) : 3167542 $\longleftrightarrow$ 13|2457|6

Why are they called fireworks ?

Grothendieck Polynomials

Grothendiecks

K-theoretic analouges of Schubert polynomials

Grothendiecks

K-theoretic analouges of Schubert polynomials

$$G_w = \begin{cases} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} & \text{if } w = w_0 \\ \overline{\partial_i}\left(G_{ws_i}(x)\right) & \text{if } w(i) < w(i+1) \end{cases}$$

Grothendiecks

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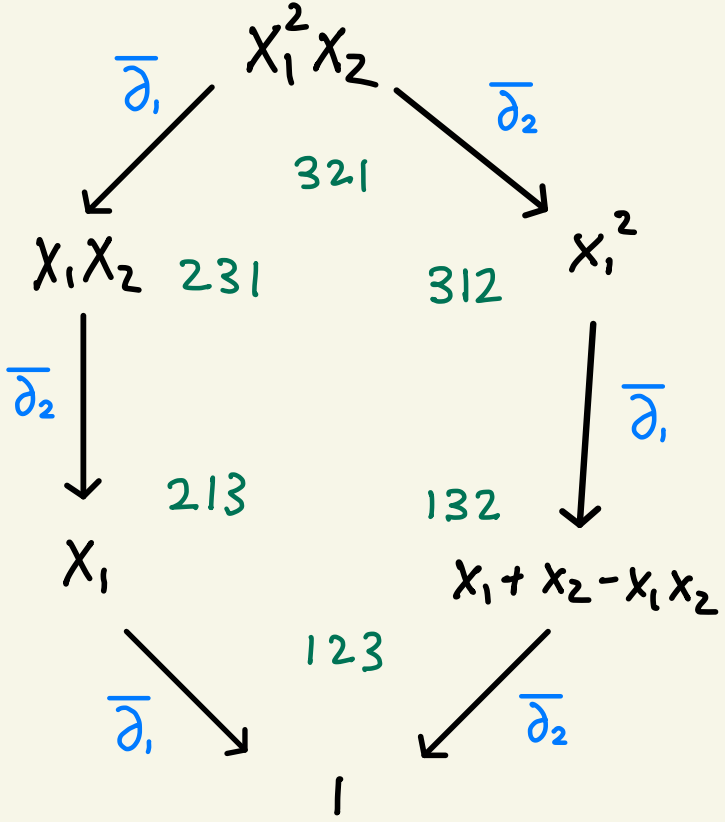
Lowest Degree : Schubert G_w

Highest Degree : Top Gro G_w^{Top}

Grothendiecks

$$\partial_i(f) = \frac{f - s_i f}{x_i - x_{i+1}}$$

$$\overline{\partial_i}(f) = \partial_i\left(f - x_{i+1} f\right)$$

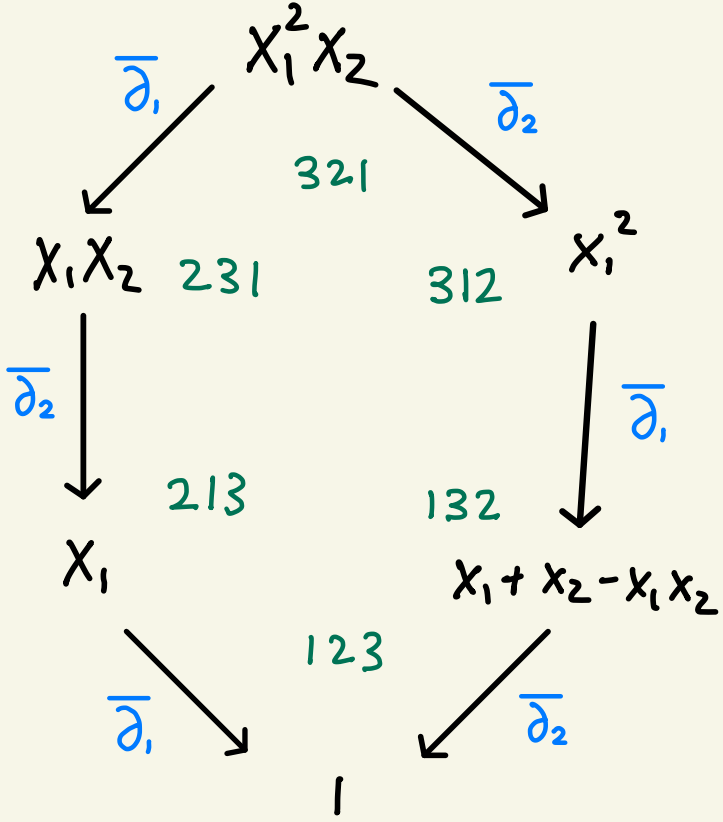


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$$G_{231} = \overline{\partial_1}\left(G_{321}\right)$$



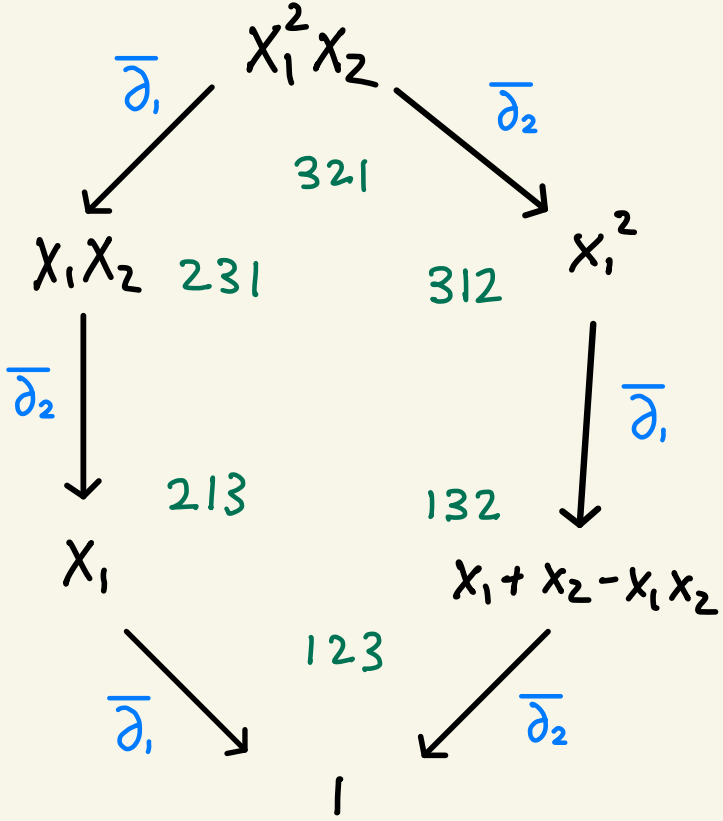
Grothendiecks

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Grothendiecks

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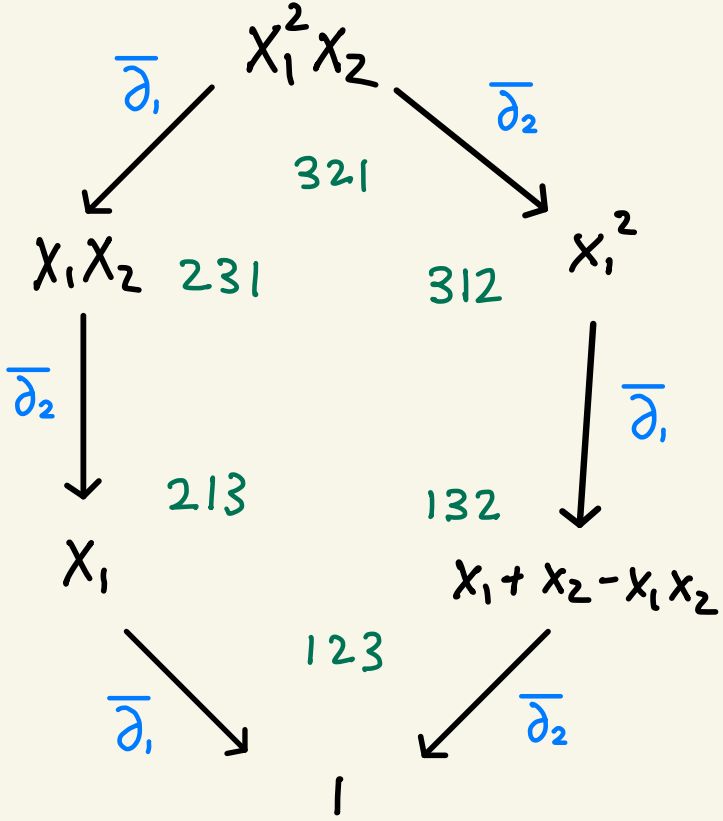
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$$= \frac{(x_1^2 x_2 - x_1^2 x_2^2) - (x_1 x_2^2 - x_1^2 x_2^2)}{x_1 - x_2}$$

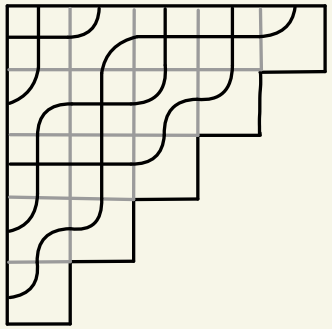
$$= x_1 x_2$$



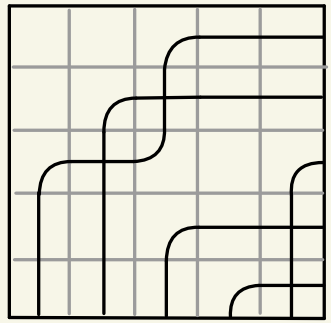
Grothendiecks

Combinatorial formulas:

Pipe Dreams and Bumpless Pipe Dreams



(Bergeron-Billey)



(Lam-Lee-Shimozono)

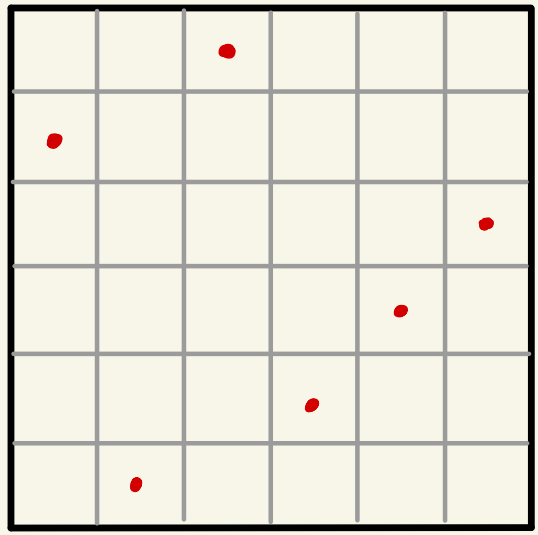
Support of Fireworks Grothendieck Polynomials

Fireworks Grothendiecks

Rothe Diagram

$D(w)$

$w = 316542$

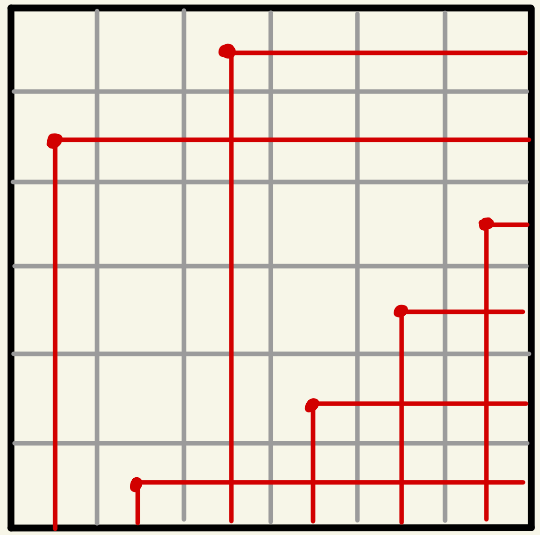


Fireworks Grothendiecks

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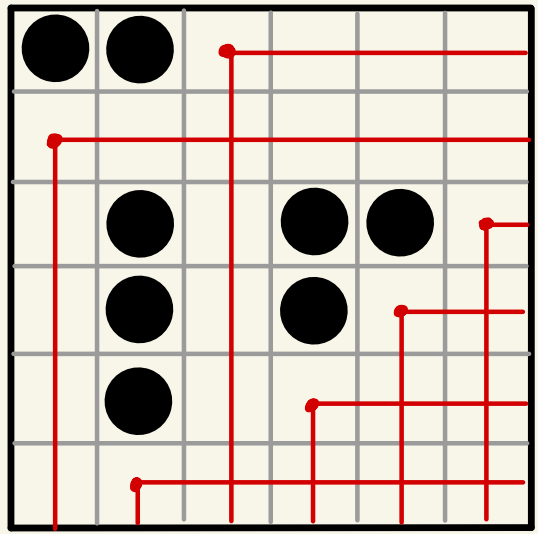


Fireworks Grothendiecks

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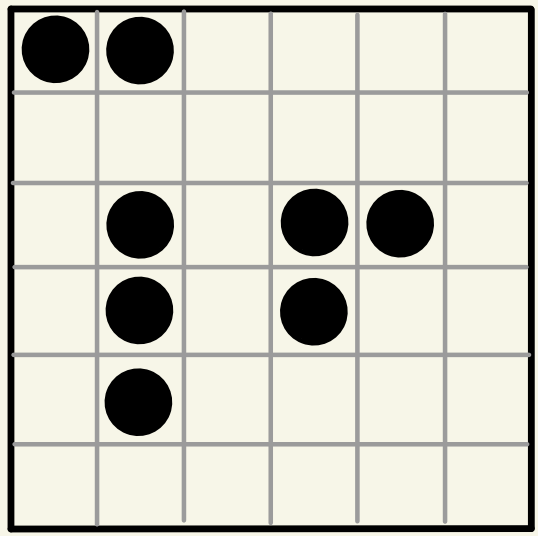
Rothe Diagram

$D(w)$

Lehmer code

$Leh(w) = [2,0,3,2,1]$

$w = 316542$

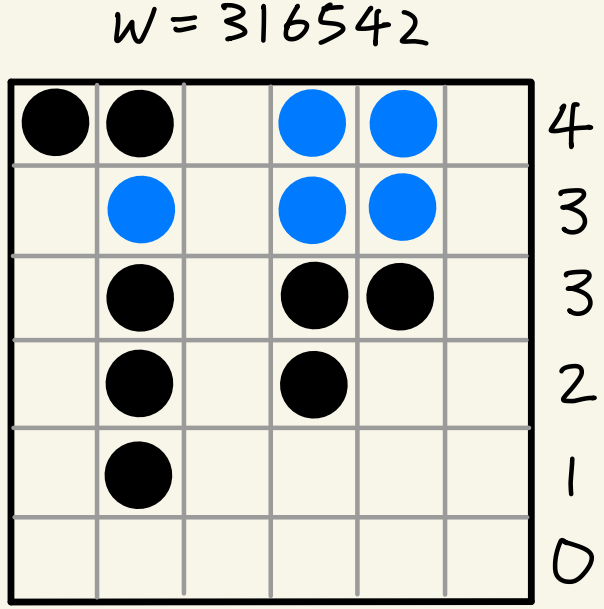


Fireworks Grothendiecks

Upward Closure of $D(w)$

$\overline{D(w)}$

$wt(\overline{D(w)}) = [4,3,3,2,1,0]$



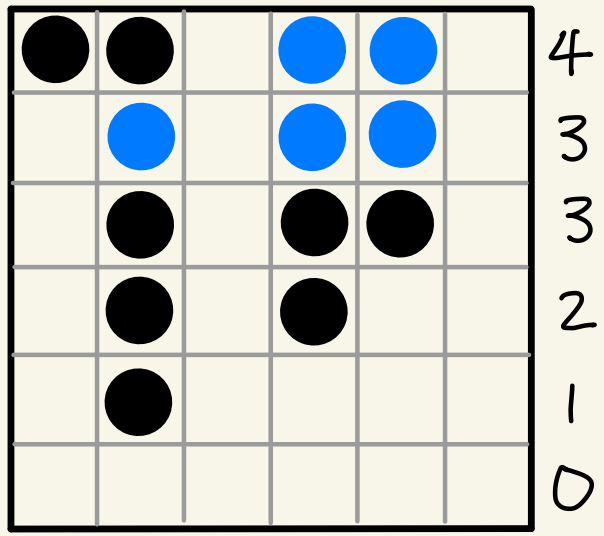
Fireworks Grothendiecks

Thm (Mészáros - Setiabrata - St. Dizier)

For $w \in S_n$ fireworks,

$$G_w^{top} = c_w \cdot X^{wt(\overline{D(w)})}$$

$w = 316542$



$$G_w^{top} = c_w \cdot X_1^4 X_2^3 X_3^3 X_4^2 X_4^1 X_5^0$$

Support

$$\mathrm{Supp}(f) = \left\{ \alpha \;\middle|\; x^{\alpha} \text{ coeff} \neq 0 \right\}$$

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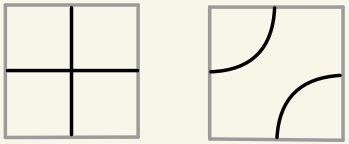
Thm (C. - Setiabrata)

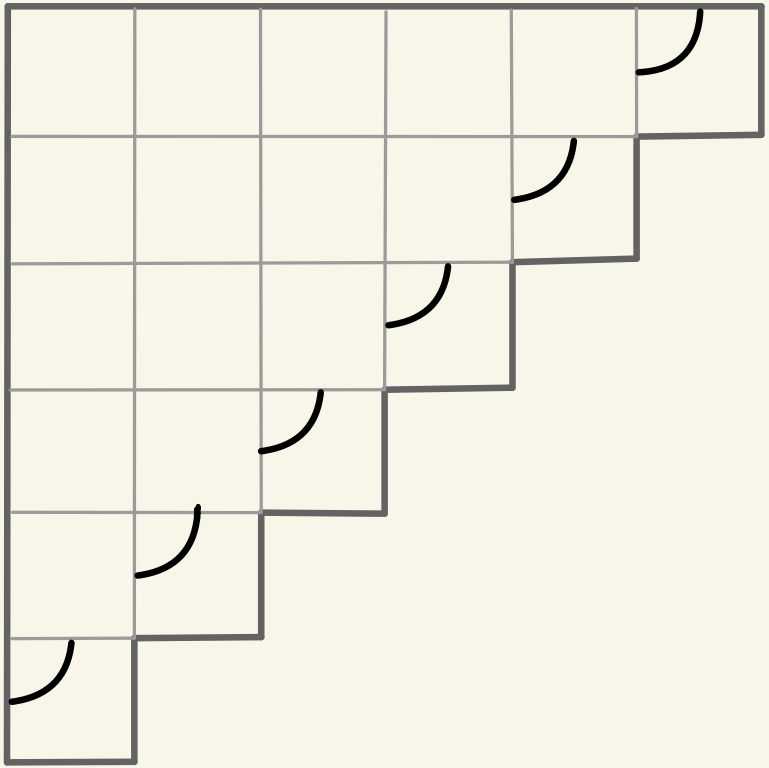
For w fireworks,

$$\mathrm{Supp}(G_w) = \bigcup_{\alpha \in \mathrm{Supp}(\mathfrak{G}_w)} [\alpha, \mathrm{wt}(\overline{D(w)})]$$

Sketch Proof

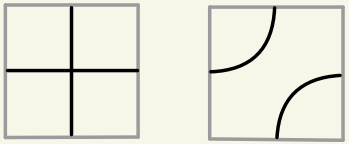
Pipe Dream

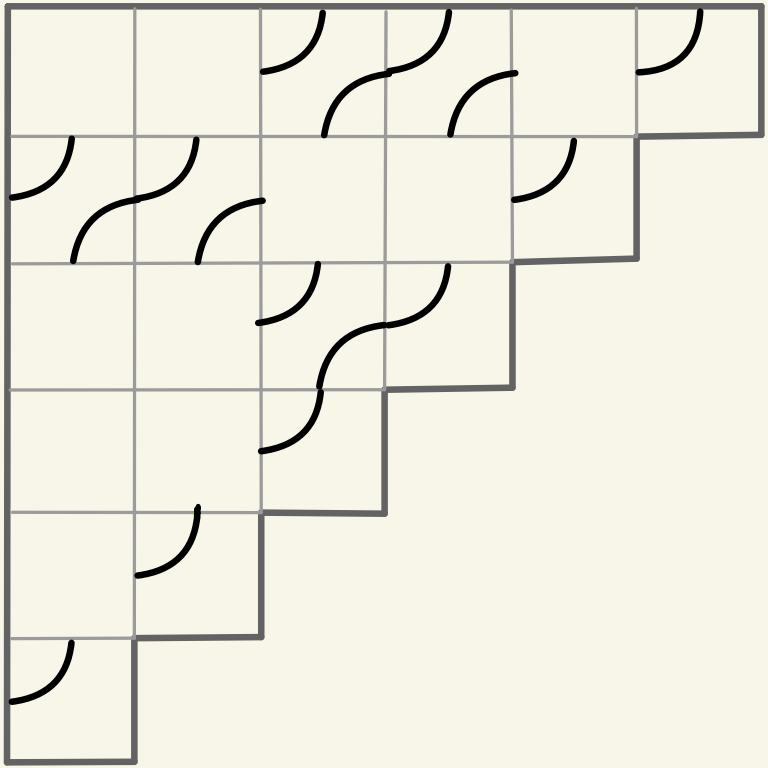




Sketch Proof

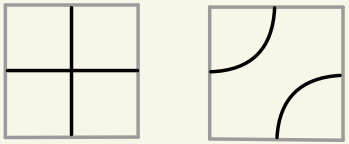
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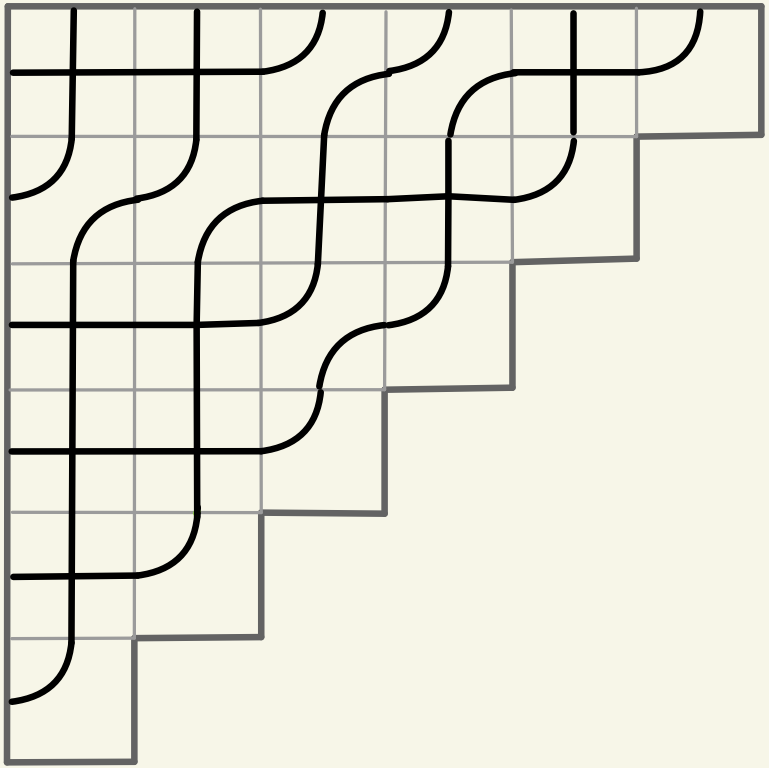




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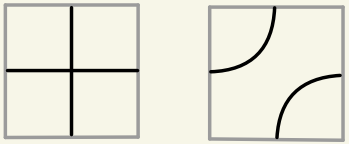
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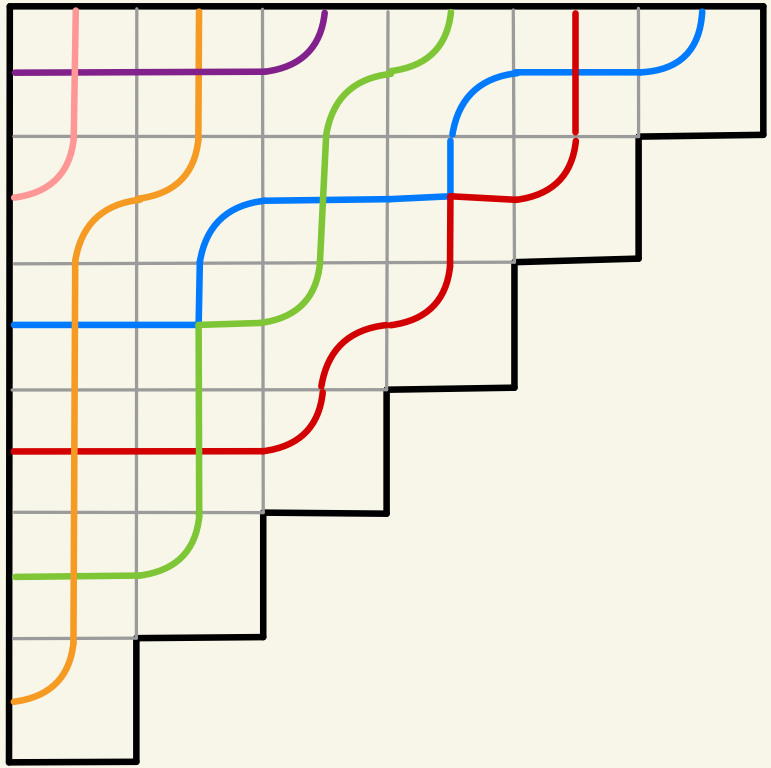
Sketch Proof

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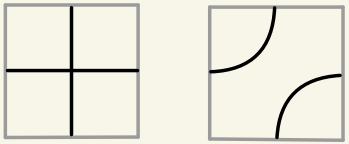
Trace pipes by

double crossing rule



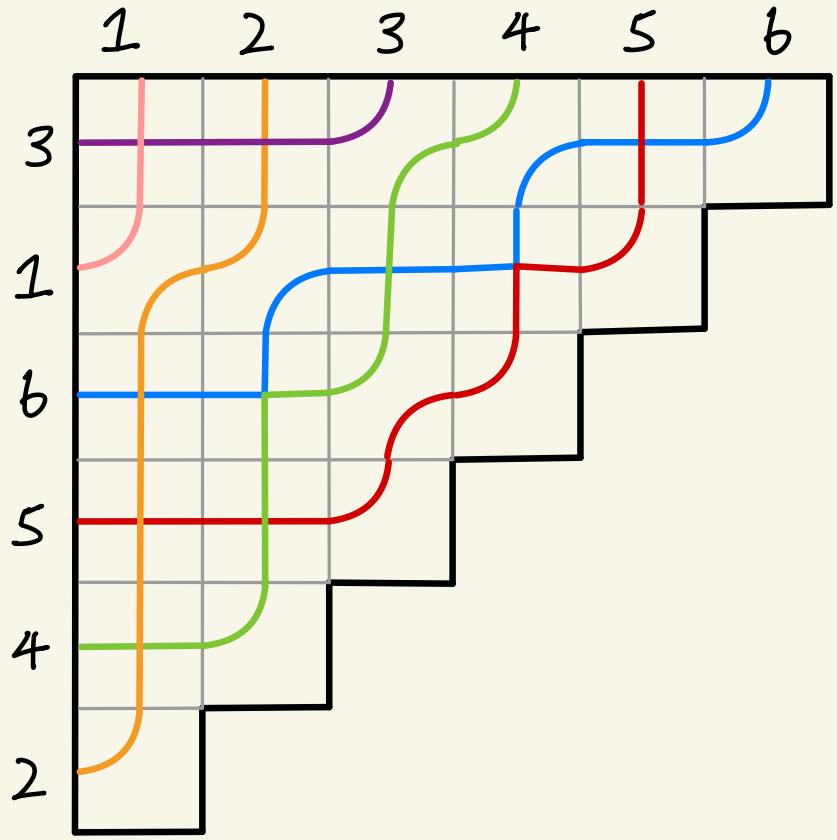
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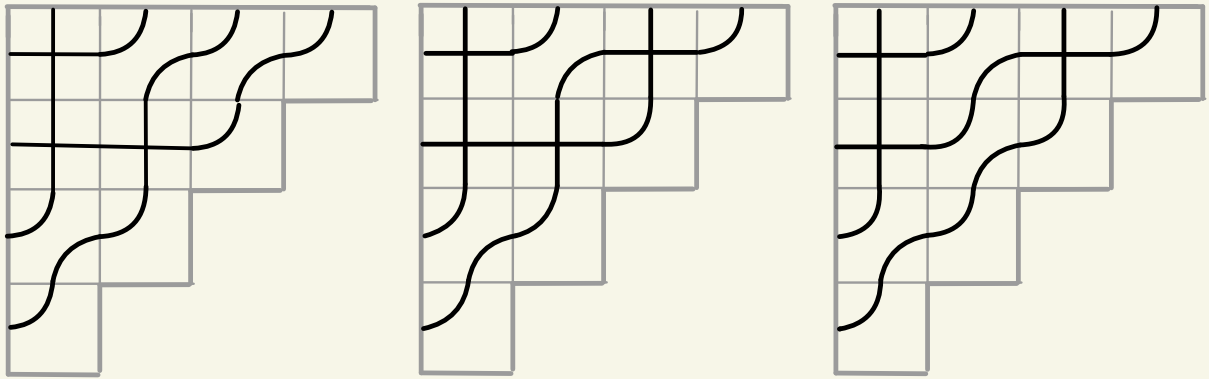
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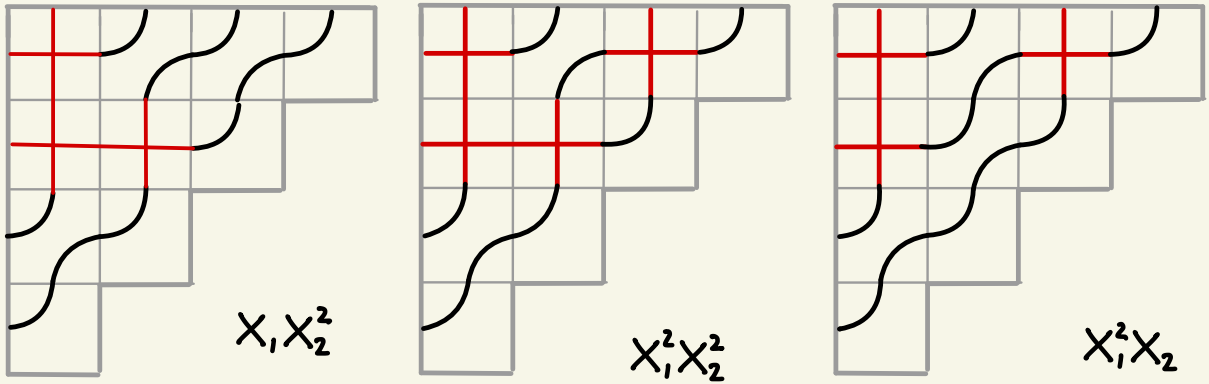
Sketch Proof

All pipe dreams of 2413



Sketch Proof

All pipe dreams of 2413



$$G_{1_{2413}} = X_1X_2^2 + X_1^2X_2 - X_1^2X_2^2$$

Sketch Proof

$$\text{Goal: } \mathrm{Supp}(G_w) = \bigcup_{\alpha \in \mathrm{Supp}(G_w)} [\alpha, \mathrm{wt}(\overline{D(w)})]$$

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Then $\exists\, \beta \in \mathrm{Supp}(G_w)$ such that

1. $\beta_j = \alpha_j \quad \forall\, j \neq i$
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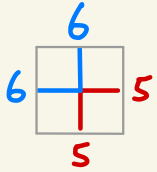
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$[2, 3, 1, 1, 3, 1]$

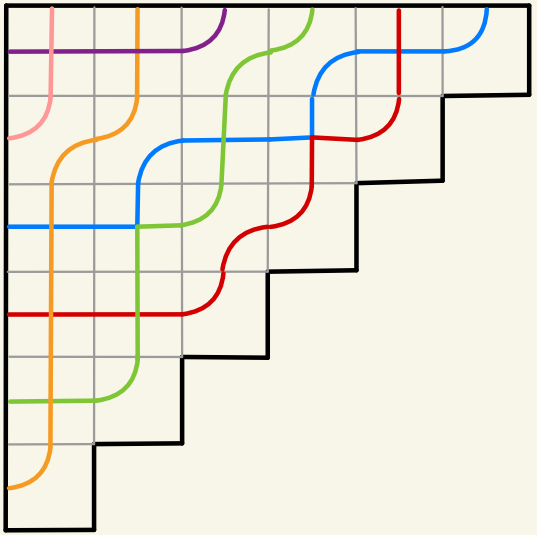
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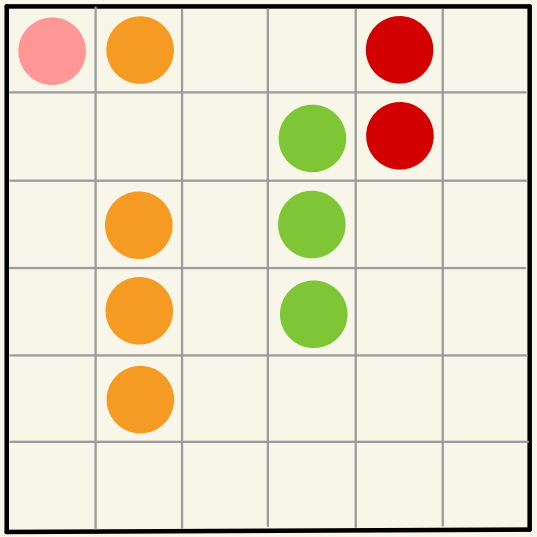
$[2, 3, 2, 1, 1, 0]$

Sketch Proof

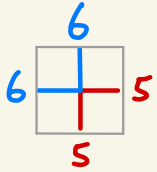
Pipe Weight: 

in row 2 $\longrightarrow$ (2, 5)

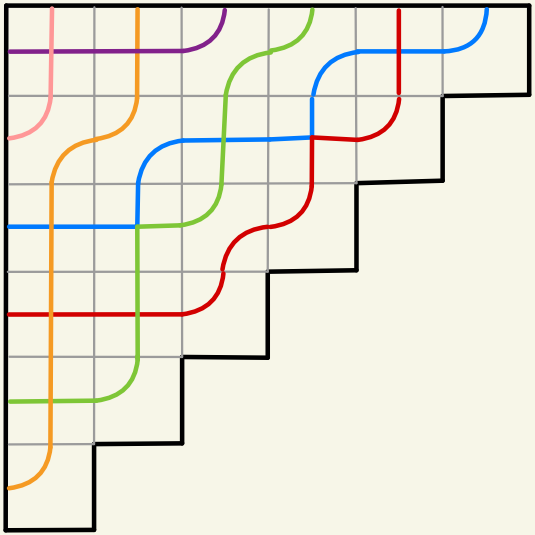


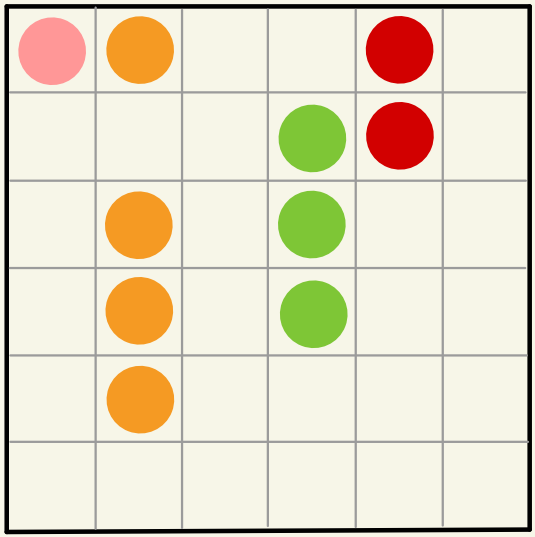


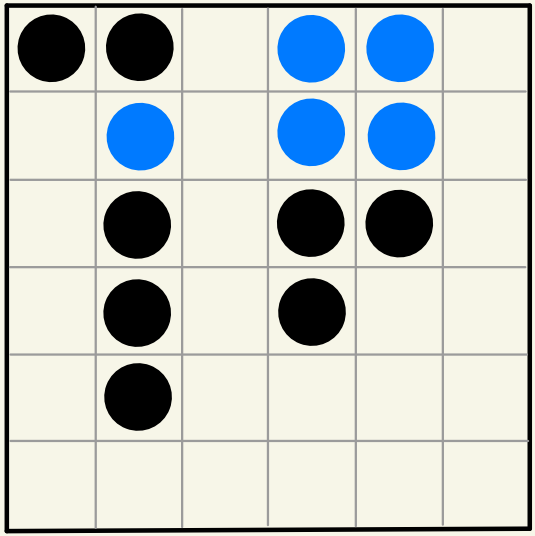
Sketch Proof

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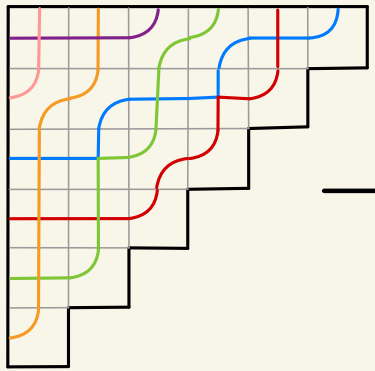




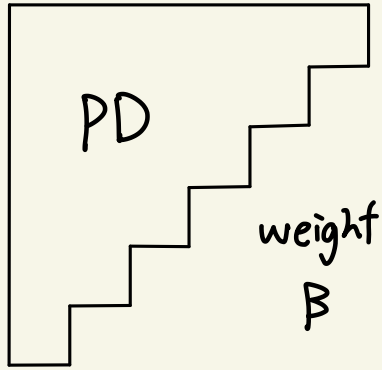
Lemma: For any pipe dream, its pipe weight is a subset of $\overline{D(w)}$.

Sketch Proof

weight α



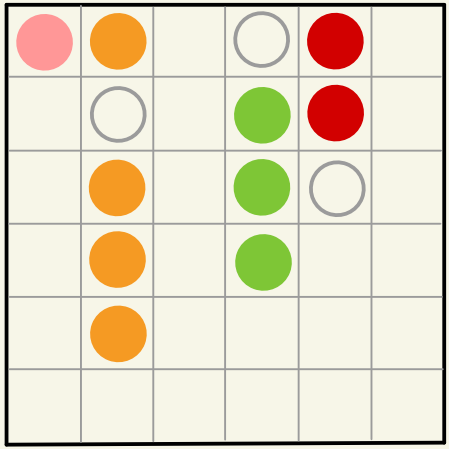
Magical Algorithm



$\beta_j = \alpha_j \quad \forall\, j < i$

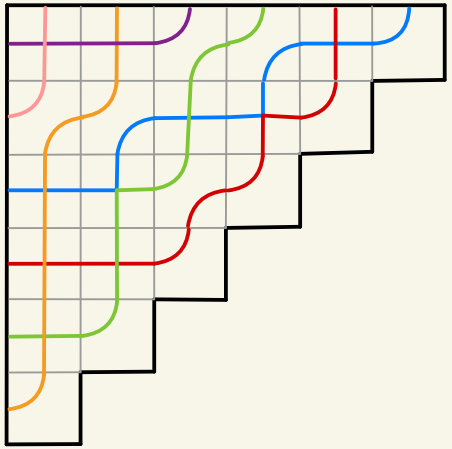
$\beta_i = \alpha_i + 1$

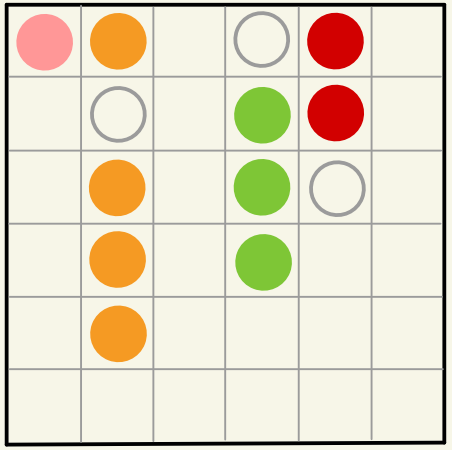
$\beta_j \leq \alpha_j \quad \forall\, j > i$



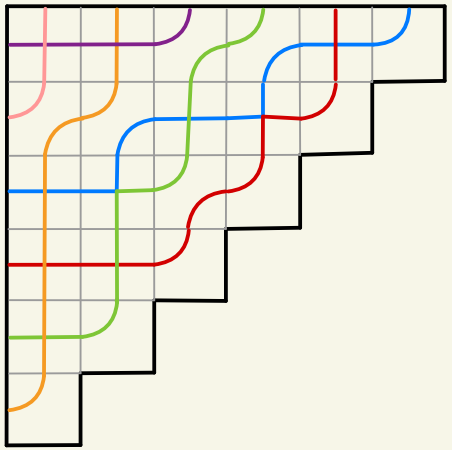
○ indicates where we can perform "magical algorithm".

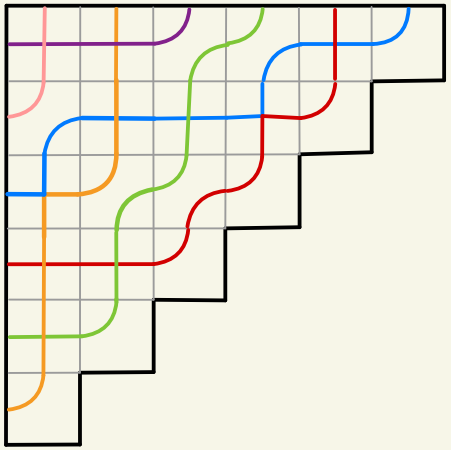
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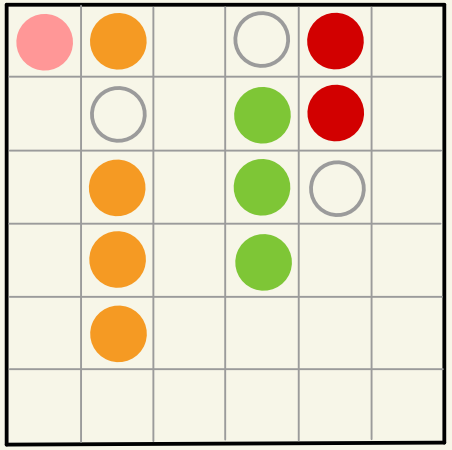


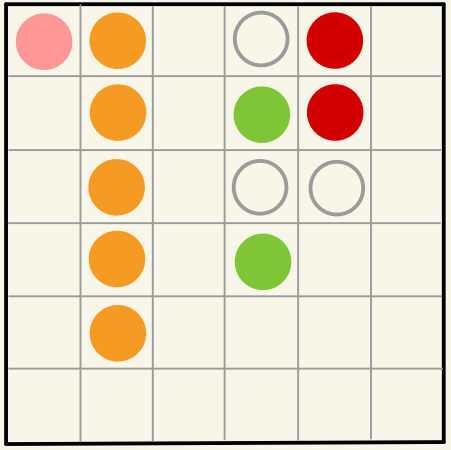


Sketch Proof

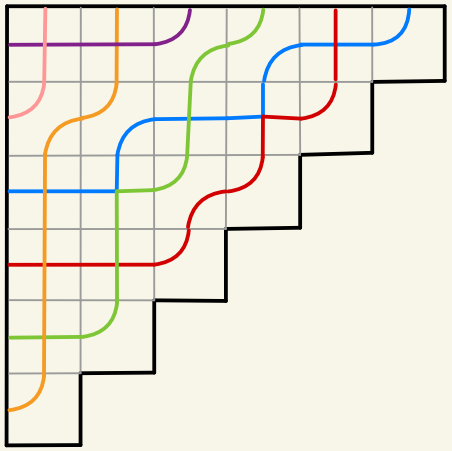


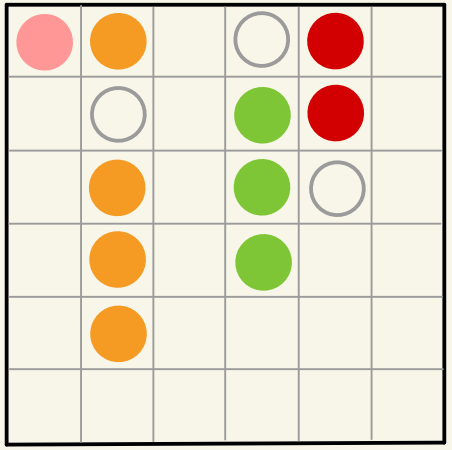


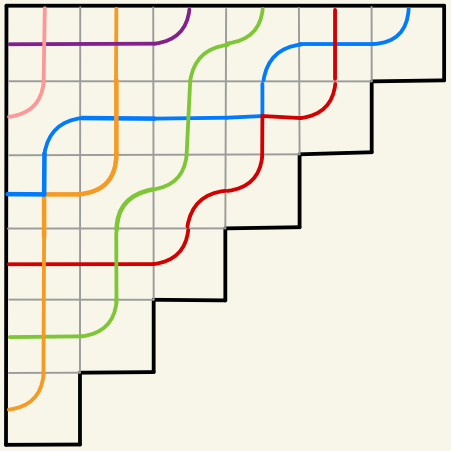


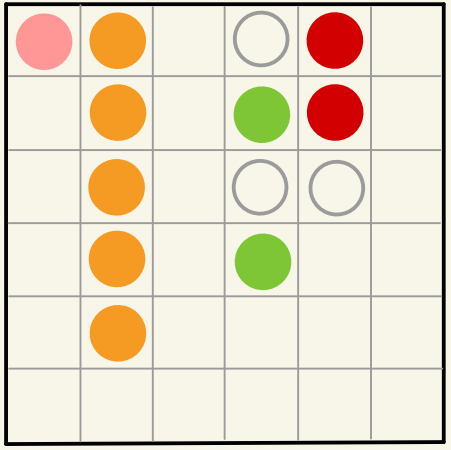


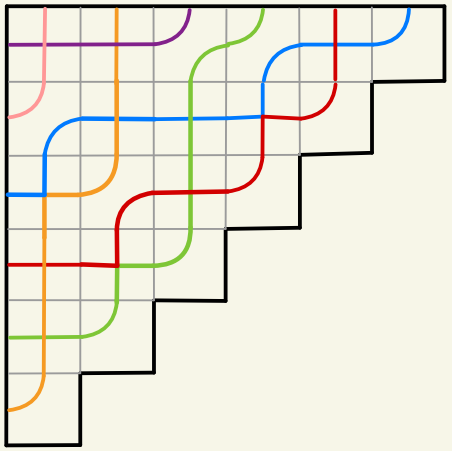
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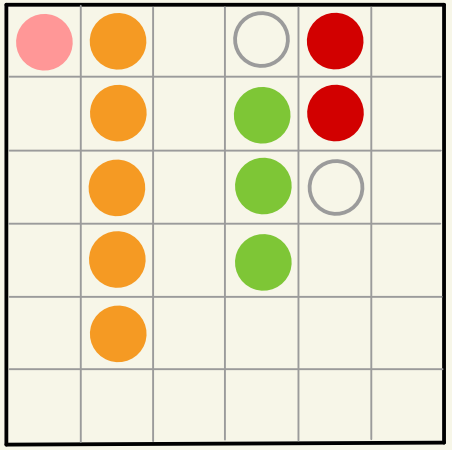












Newton Polytopes

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Conj. (Monical - Tokcan - Yong)

G_w has SNP.

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For w fireworks, G_w has SNP.

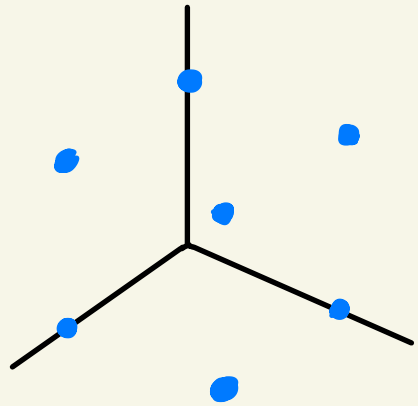
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$$G_w = X_1 + X_2 + X_3 - X_1X_2 - X_1X_3 - X_2X_3 + X_1X_2X_3$$



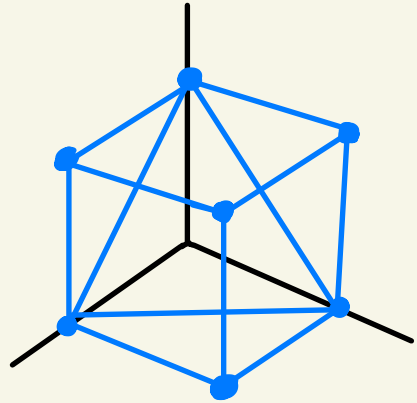
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$$f \text{ Lorentzian} = \underset{\text{M-convex}}{\text{Supp}(f)} + \underset{\text{on coefficients}}{\text{certain log-concavity}}$$

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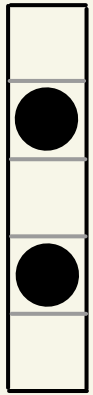
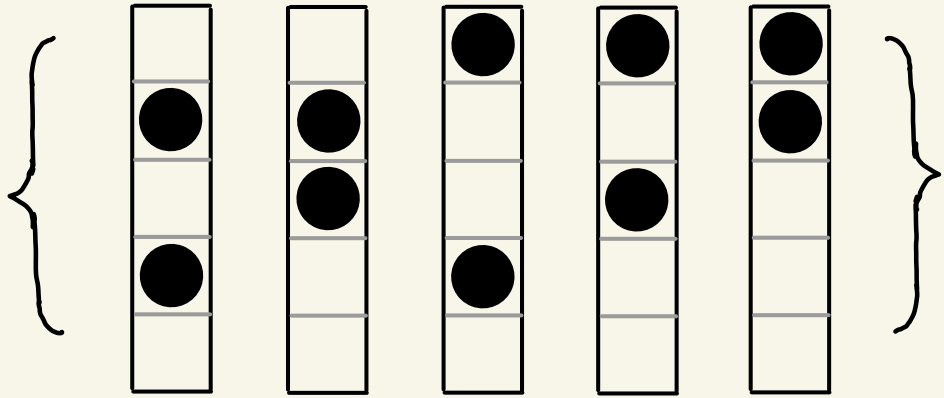
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$$\text{M-convexity} = \text{SNP} + \text{Generalized Permutahedron}$$

M-Convexity

Schubert Matroid Polytopes

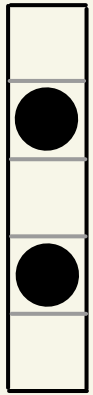
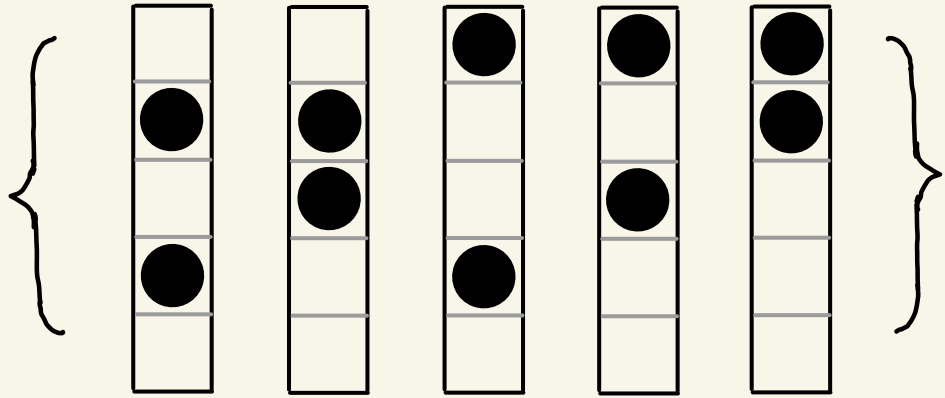
$S = \{2,4\} \subseteq [5]$

$SM($  $) =$ 

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Schubert Matroid Polytopes

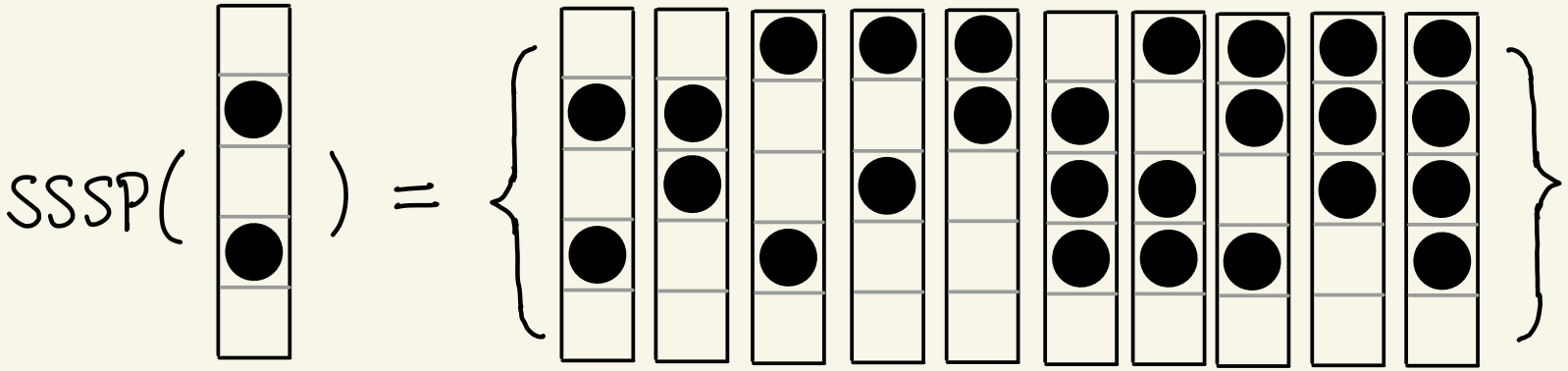
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$P(SM(S))$ is SNP and Generalized Permutahedron

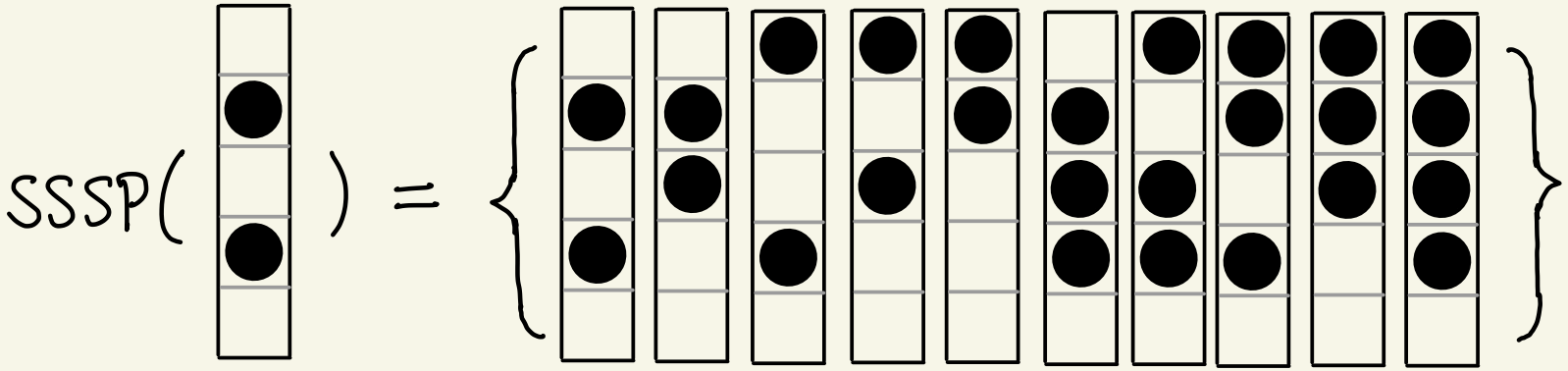
M-Convexity

Schubert Spanning Set Polytopes (SSSP)



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Minkowski Sum $P + Q = \{p+q \mid p \in P,\ q \in Q\}$

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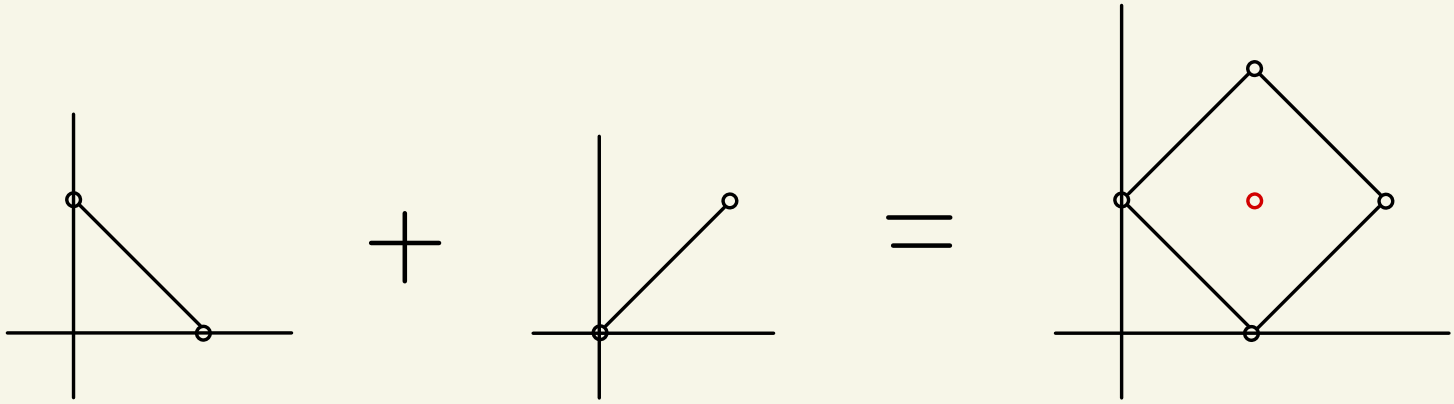
2) Gen. Perm. + Gen. Perm. = Gen. Perm.

3) $Supp(f) + Supp(g) \neq$ SNP

SNP SNP

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Minkowski Sum $P + Q = \{p+q \mid p \in P,\ q \in Q\}$



3) $\underset{\text{SNP}}{\text{Supp}(f)} + \underset{\text{SNP}}{\text{Supp}(g)} \neq \text{SNP}$

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Minkowski Sum $P + Q = \{p+q \mid p \in P,\ q \in Q\}$

Generalized Permutahedron: Integer Decomposition Property

If $\alpha \in (P_1 + P_2) \cap \mathbb{Z}^n$, then $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in P_i \cap \mathbb{Z}^n$

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M-convex + M-convex = M-convex

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Thm. (Fink-Mészáros-St. Dizier)

$$\mathrm{Supp}(G_w) = \sum P(SM(D(w)_i)) \cap \mathbb{Z}^n$$

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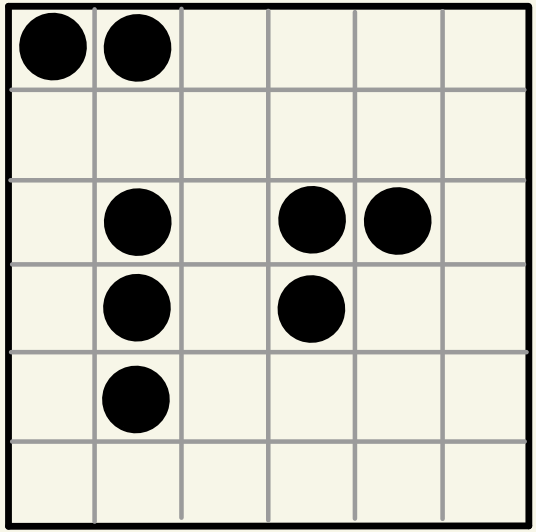
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Cor. Fireworks Grothendieck polynomials have M-convex support.

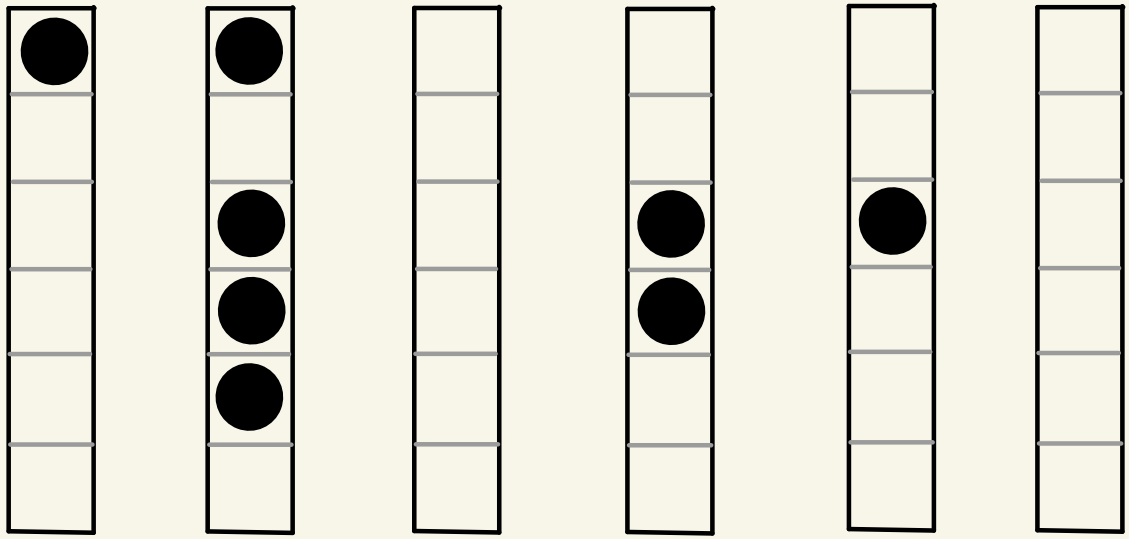
M- Convexity

D(316542)



M- Convexity

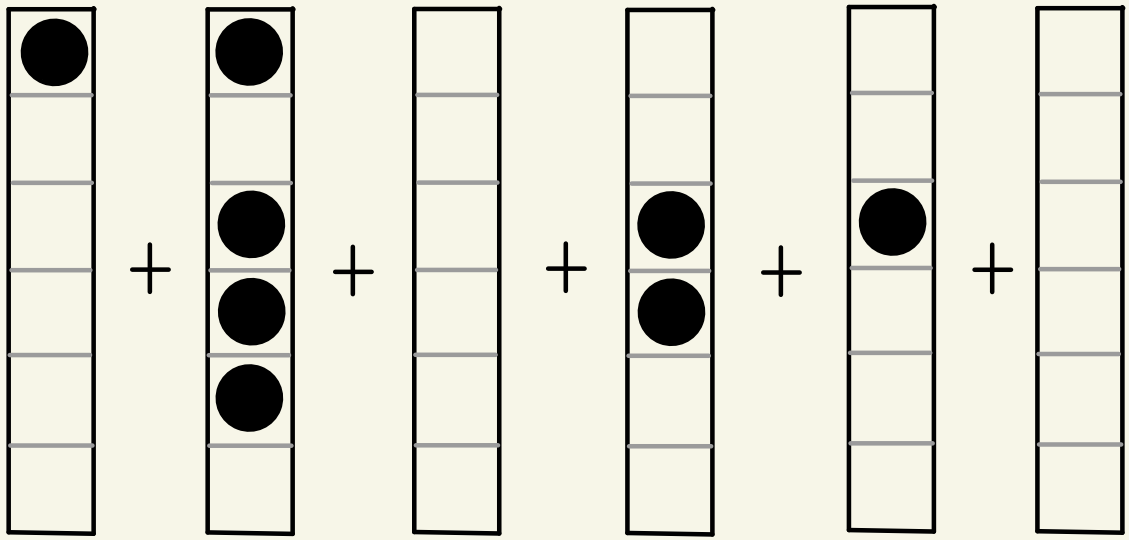
$w = 316542$



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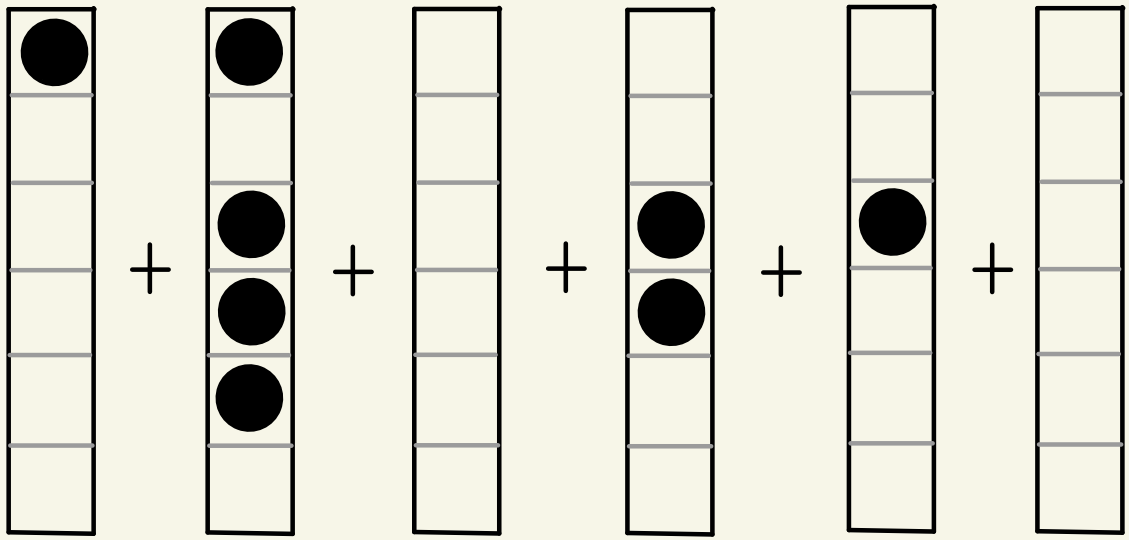
$$\sum \text{Schubert Matroid Polytopes} = \text{Newton}(\mathfrak{S}_w)$$



M- Convexity

$w = 316542$

$\sum$ Schubert Spanning Set Polytopes = Newton(G_w)



Layered Permutations

Layered

Layered = Direct Sum of Long Elements

Layered

Layered = Direct Sum of Long Elements

= Fireworks + Inverse Fireworks

Layered

Layered = Direct Sum of Long Elements

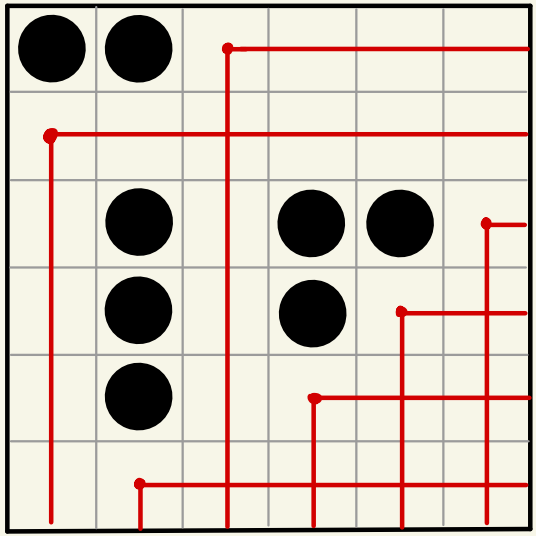
= Fireworks + Inverse Fireworks

Thm. (Pechenik-Speyer-Weigandt)

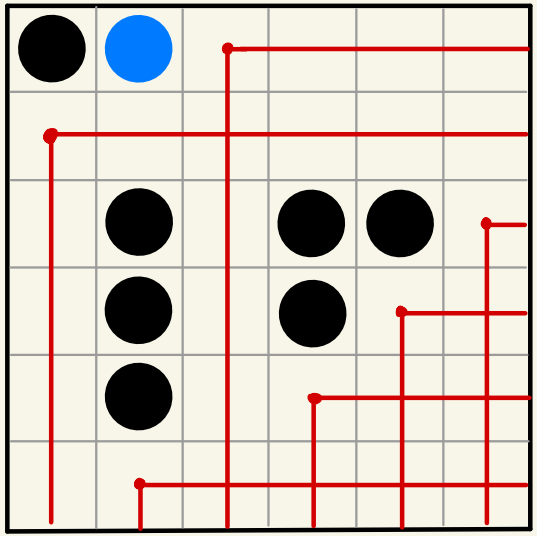
$\forall w \in S_n$, $\exists!$ inverse fireworks permutation $\pi(w)$

such that $G_w^{top} = c_w \cdot G_{\pi(w)}^{top}$.

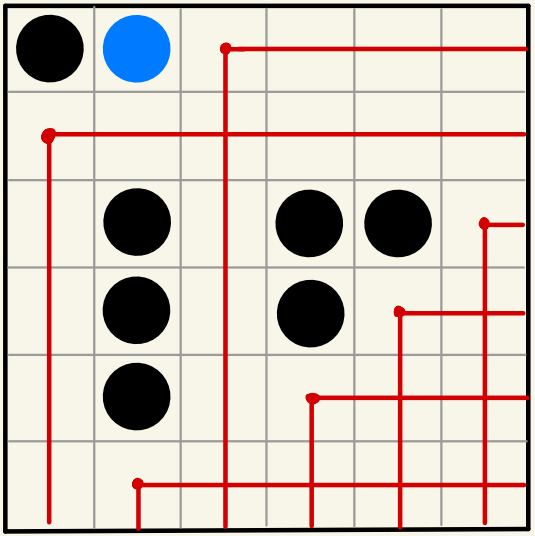
Layered

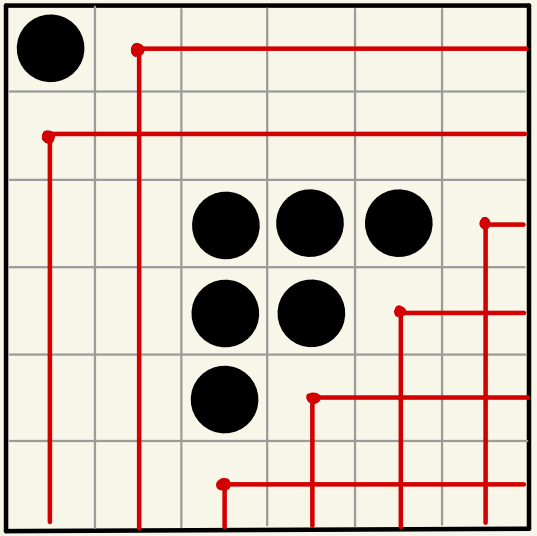


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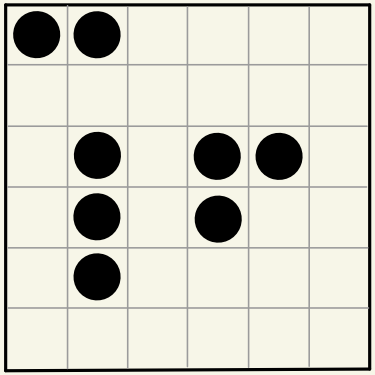


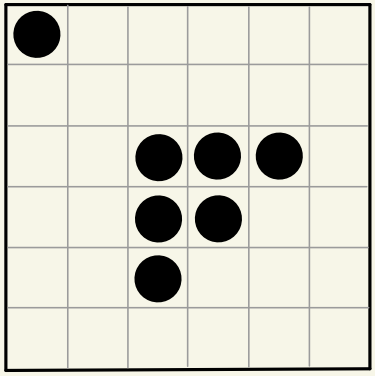
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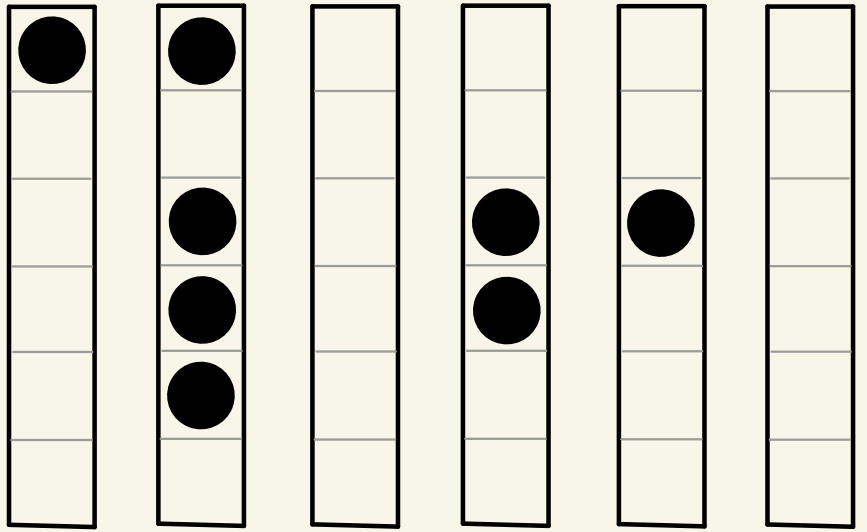


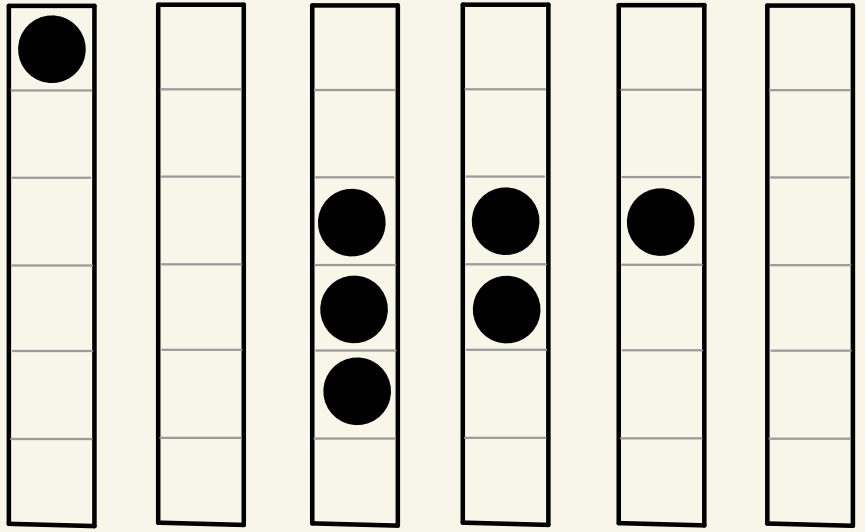


Layered









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Layered

Conj. (Merzon - Smirnov)

Layered permutations achieve max of $G_w(1,\cdots,1)$.

Thm. (Morales - Panova - Petrov - Yeliussizov)

Layered permutations asymptotically achieve max of $|G_w(-1,\cdots,-1)|$.

Thm. (Pechenik - Speyer - Weigandt)

Layered permutations maximizes $\deg(G_w) - \deg(\mathfrak{G}_w)$.

Conj. Layered permutations have largest support.

Thank You !!

Thm. For w fireworks,

1. $\mathrm{Supp}(G_w) = \bigcup_{\alpha \in \mathrm{Supp}(G_w)} [\alpha, \mathrm{wt}(\overline{D(w)})]$
2. G_w have M-convex support.
3. $\pi(w)$ is layered and $\mathrm{Supp}(G_w) \subseteq \mathrm{Supp}(G_{\pi(w)})$

