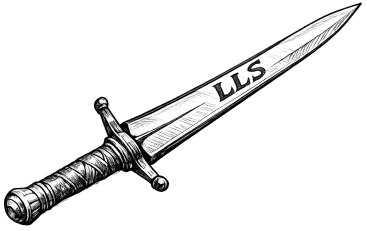
A positive combinatorial formula for the double Edelman-Greene coefficients

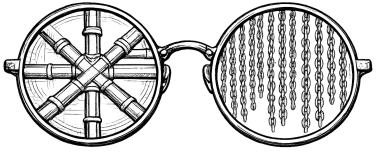
Jack Chou

joint with Tianyi Yu

Outline:

- Lam, Lee, and Shimozono defined the double Edelman-Greene coefficients $\bar{J}^{w}_{\lambda}(y)$.
- They showed that $\bar{J}^{w}_{\lambda}(y)$ are "positive" geometrically, and asked for combinatorial proof.
- We solve this problem using 3 weapons:







(Double) Schubert Polynomials

Represent classes in T-equivarient cohomology of flag varieties.

$$\mathfrak{S}_{2143} = (x_1-y_1)(x_3-y_3) + (x_1-y_1)(x_1-y_2) + (x_1-y_1)(x_2-y_1)$$

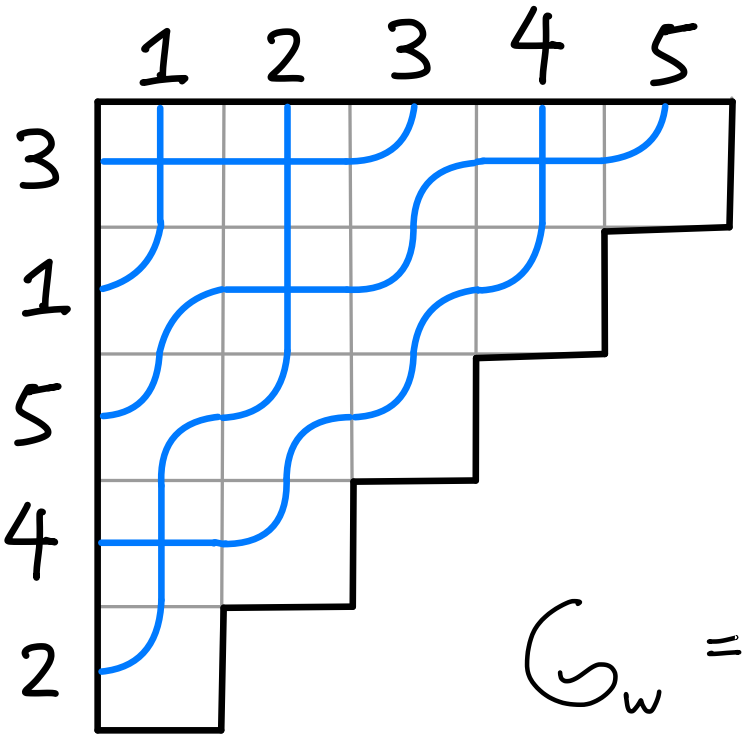
2143: permutation

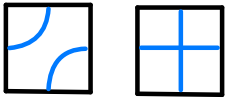
$(x_1-y_1)(x_1-y_2)$: product of (x_i-y_j)

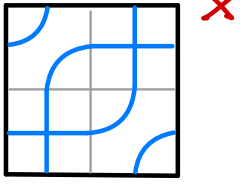
Can be defined recursively (divided difference)

Have many combinatorial formulae

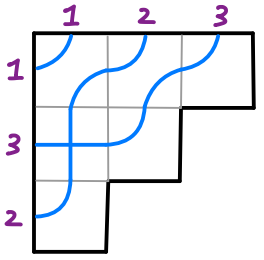
Pipe Dreams: [Bergeron-Billey] [Billey-Jockusch-Stanley]

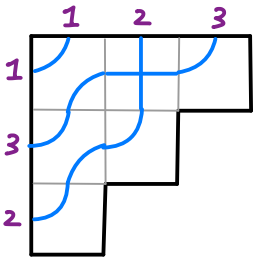


Tiles 

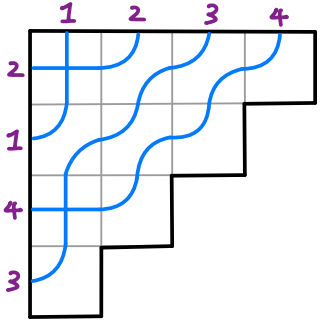
No double crossings 

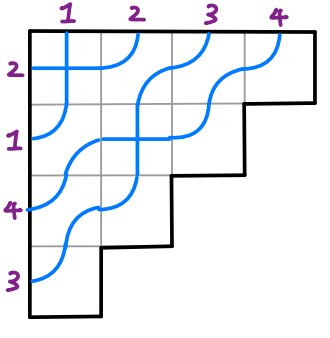
$$\mathfrak{S}_w = \sum_{D \in PD(w)} \prod_{(i,j) = \square} (x_i - y_j)$$

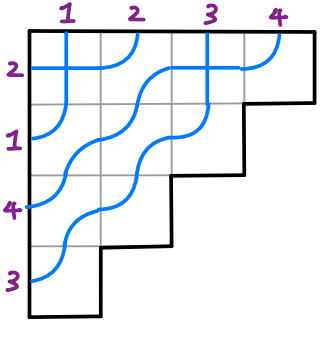




$$G_{132} = (x_2 - y_1) + (x_1 - y_2)$$

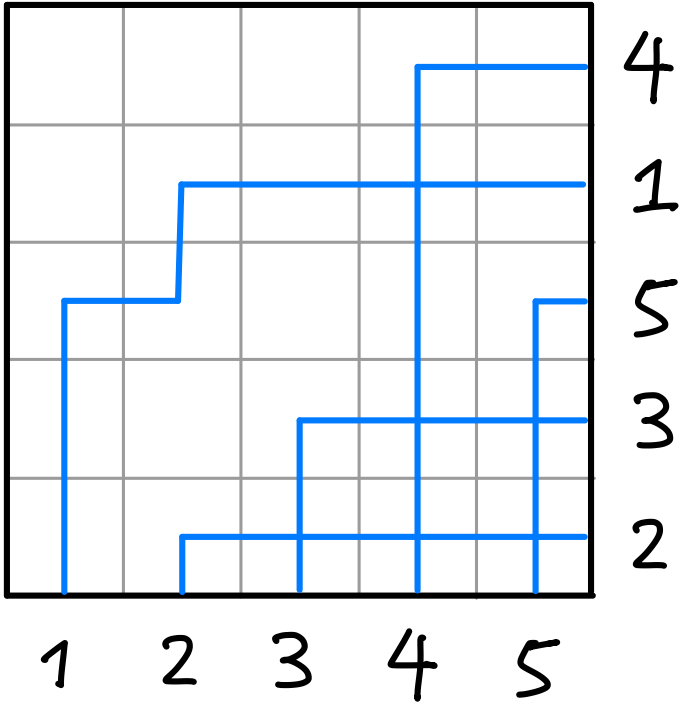




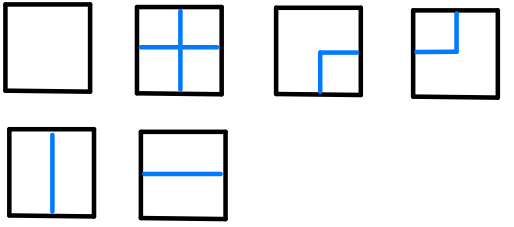


$$G_{2143} = (x_1 - y_1)(x_3 - y_1) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_3)$$

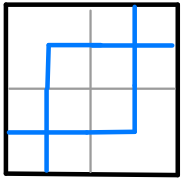
Bumpless Pipe Dreams: [Lam-Lee-Shimozono]



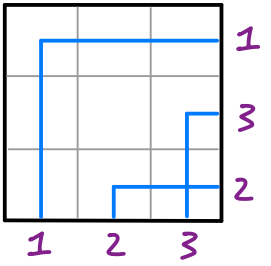
Tiles

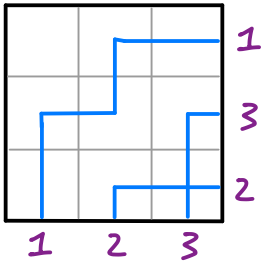


No double crossings

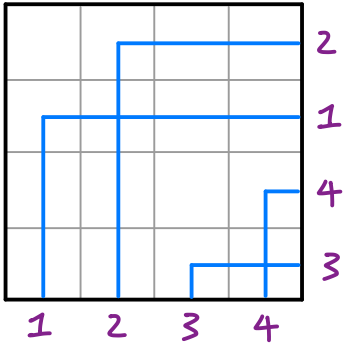


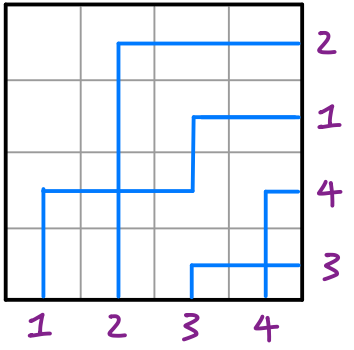
$$\mathfrak{S}_w = \sum_{D \in BPD(w)} \prod_{(i,j)=\square} (x_i - y_j)$$

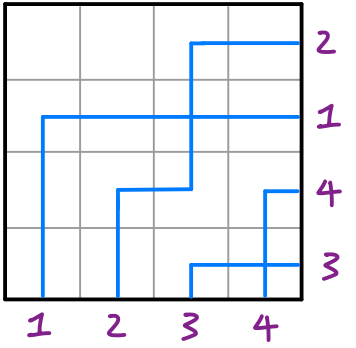




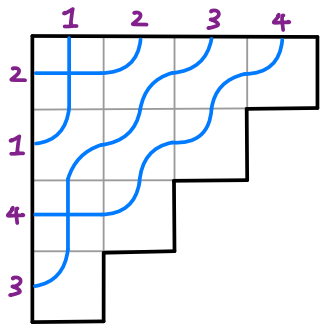
$$\mathcal{G}_{132} = (x_2 - y_2) + (x_1 - y_1)$$

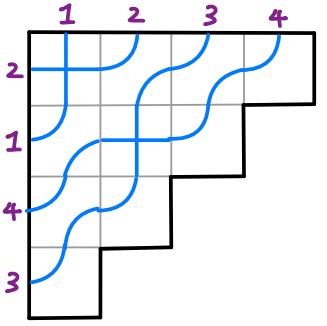


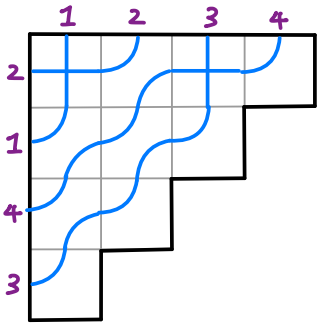




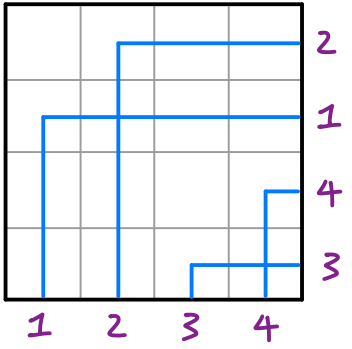
$$\mathcal{G}_{2143} = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2)$$

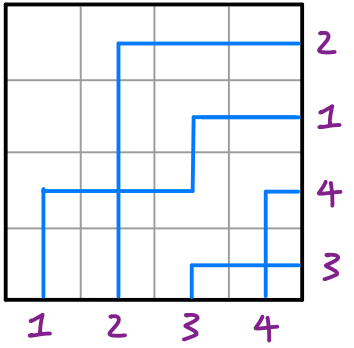


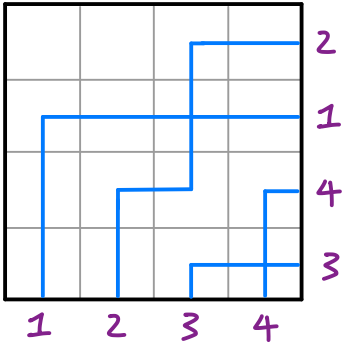




$$\mathfrak{S}_{2143} = (x_1-y_1)(x_3-y_1) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_3)$$







$$\mathfrak{S}_{2143} = (x_1-y_1)(x_3-y_3) + (x_1-y_1)(x_2-y_1) + (x_1-y_1)(x_1-y_2)$$

Backstable G_w [Lam-Lee-Shimozono]

$$G_w(x_1, x_2, \cdots; y_1, y_2, \cdots)$$

$$G_{1\times w}(x_0, x_1, x_2, \cdots; y_0, y_1, y_2, \cdots)$$

$$G_{1^2\times w}(x_{-1}, x_0, x_1, x_2, \cdots; y_{-1}, y_0, y_1, y_2, \cdots)$$

$$\vdots$$

Infinitely many times!

$$\overleftarrow{G_w}(\cdots, x_{-1}, x_0, x_1, x_2, \cdots; \cdots, y_{-1}, y_0, y_1, y_2, \cdots)$$

Backstable $\mathfrak{S}_w$ [Lam-Lee-Shimozono]

$$\mathfrak{S}_{132} = (x_2 - y_2) + (x_1 - y_1)$$

$$\mathfrak{S}_{1\times w}(x_0, x_1, x_2, \cdots ; y_0, y_1, y_2, \cdots)$$

$$\mathfrak{S}_{1^2\times w}(x_{-1}, x_0, x_1, x_2, \cdots ; y_{-1}, y_0, y_1, y_2, \cdots)$$

$$\vdots$$

Infinitely many times!

$$\overleftarrow{\mathfrak{S}_w}(\cdots, x_{-1}, x_0, x_1, x_2, \cdots ; \cdots, y_{-1}, y_0, y_1, y_2, \cdots)$$

Backstable $\mathfrak{S}_w$ [Lam-Lee-Shimozono]

$$\mathfrak{S}_{132} = (x_2 - y_2) + (x_1 - y_1)$$

$$\mathfrak{S}_{1243} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0)$$

$$\mathfrak{S}_{1^2 \times w}(x_{-1}, x_0, x_1, x_2, \cdots ; y_{-1}, y_0, y_1, y_2, \cdots)$$

$$\vdots$$

Infinitely many times!

$$\overleftarrow{\mathfrak{S}_w}(\cdots, x_{-1}, x_0, x_1, x_2, \cdots ; \cdots, y_{-1}, y_0, y_1, y_2, \cdots)$$

Backstable $\mathfrak{G}_w$ [Lam-Lee-Shimozono]

$$\mathfrak{G}_{132} = (x_2 - y_2) + (x_1 - y_1)$$

$$\mathfrak{G}_{1243} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0)$$

$$\mathfrak{G}_{12354} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0) + (x_{-1} - y_{-1})$$

$$\vdots$$

Infinitely many times!

$$\overleftarrow{\mathfrak{G}}_w(\cdots, x_{-1}, x_0, x_1, x_2, \cdots; \cdots, y_{-1}, y_0, y_1, y_2, \cdots)$$

Backstable $\mathfrak{S}_w$ [Lam-Lee-Shimozono]

$$\mathfrak{S}_{132} = (x_2 - y_2) + (x_1 - y_1)$$

$$\mathfrak{S}_{1243} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0)$$

$$\mathfrak{S}_{12354} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0) + (x_{-1} - y_{-1})$$

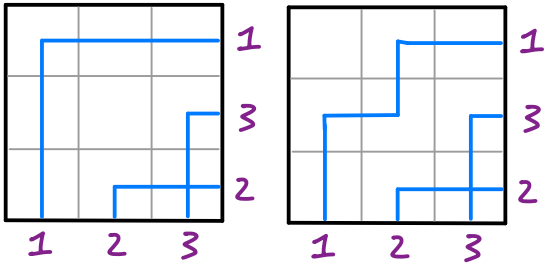
$$\vdots$$

Infinitely many times!

$$\overleftarrow{\mathfrak{S}}_{132} = (x_2 - y_2) + (x_1 - y_1) + \sum_{i \leq 0} (x_i - y_i)$$

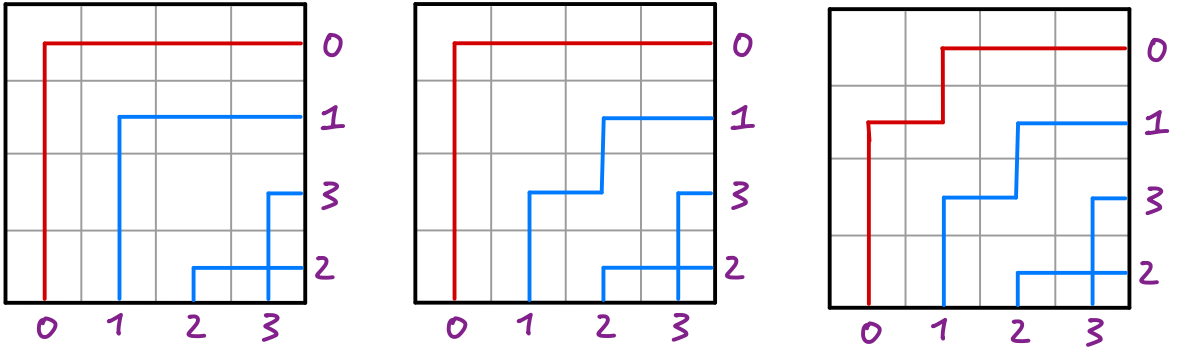
Backstable $\mathfrak{S}_w$

$$\mathfrak{S}_{132} = (x_2 - y_2) + (x_1 - y_1)$$

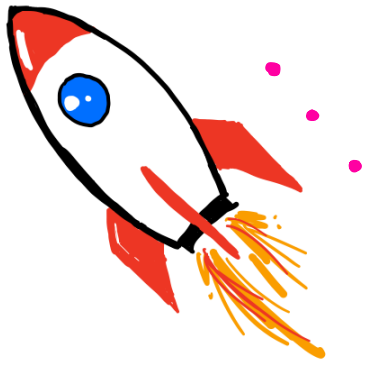


$$\mathfrak{S}_{1243} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0)$$

"0132"

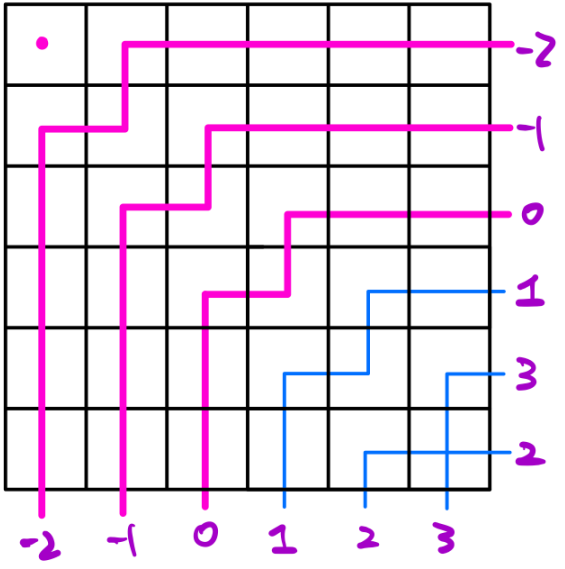


Backstable $\subset_w$



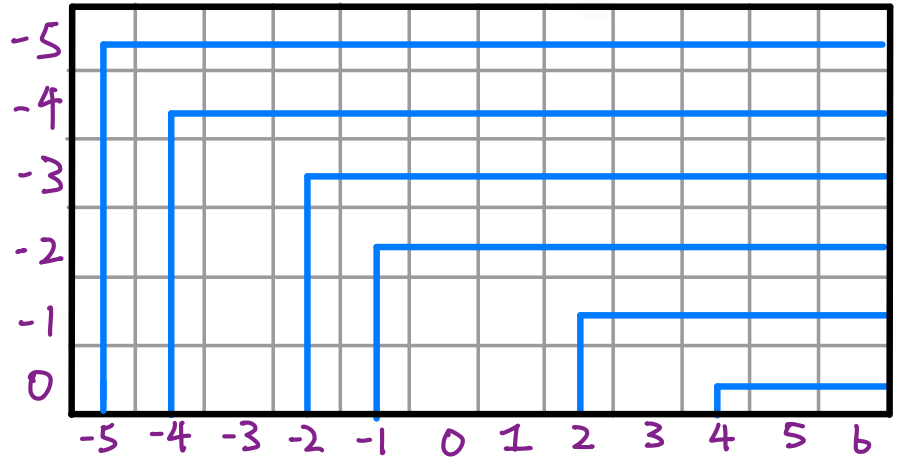
To Infinity and Beyond!

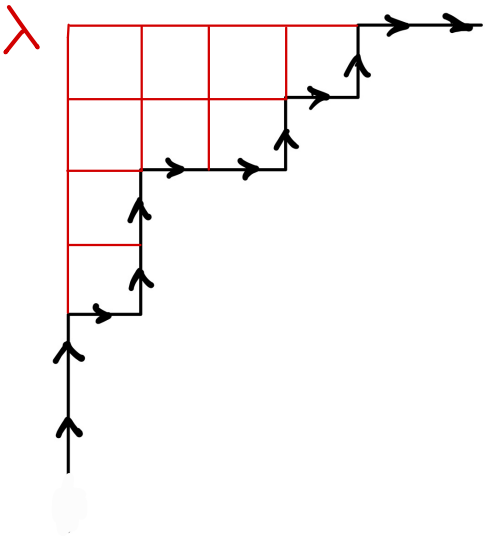
$$\overleftarrow{\subset}_{132} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0) + (x_{-1} - y_{-1}) + (x_{-2} - y_{-2}) + \cdots$$



Double Schur $S_\lambda(x;y)$

- Introduced by Molev
- Becomes $S_\lambda(x_0, x_{-1}, x_{-2}, \cdots)$ when you set $y \mapsto 0$
- Has a tableaux formula
- Has a BPD formula





Double EG Coeff:

$$\overleftarrow{G}_w = \sum_\lambda a_\lambda^w(x;y)\, s_\lambda(x;y)$$

$a_\lambda^w(x;y)$: Coefficients in $\mathbb{Q}[x;y]$

$s_\lambda(x;y)$: Double Schur

Double Edelman-Greene coefficients:

$$\overline{J}_\lambda^w(y) := a_\lambda^w(y;y) \qquad \text{Set } x_i \mapsto y_i$$

Double EG Coeff:

Theorem [Lam-Lee-Shimozono]

$$j_\lambda^w(y) \in \mathbb{Z}_{\geq 0}[\, y_i - y_j : i \prec j \,]$$

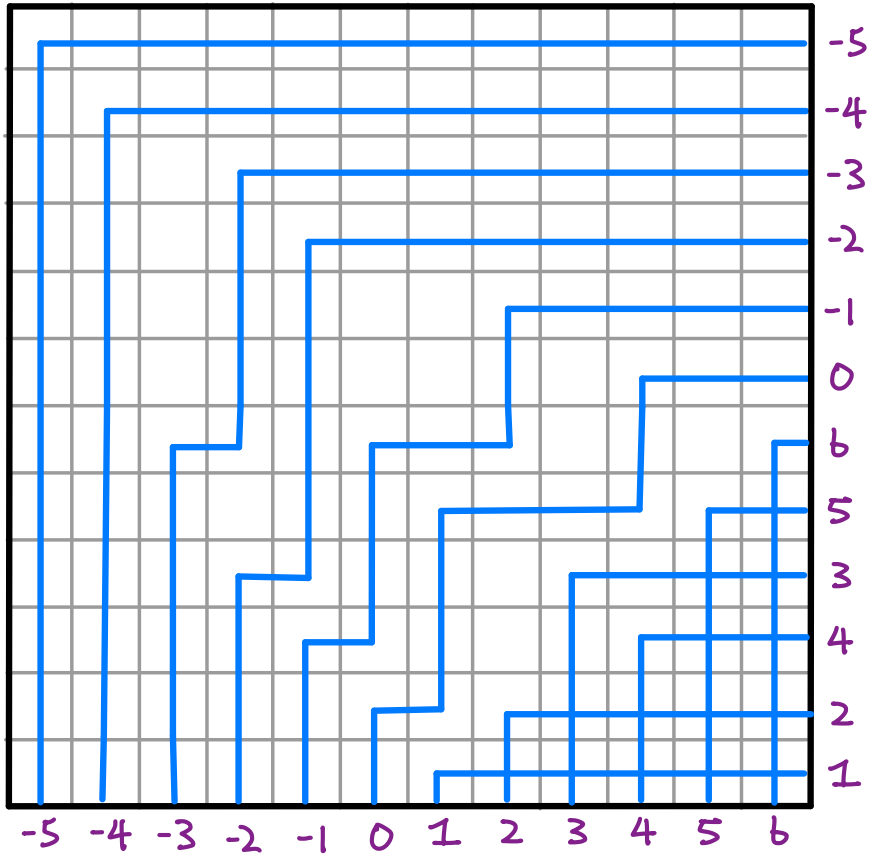
$$1 \prec 2 \prec 3 \prec \cdots \prec -2 \prec -1 \prec 0$$

Positive

Ex. $2(y_1 - y_4)^2 (y_3 - y_{-1}) + (y_{-1} - y_0)$

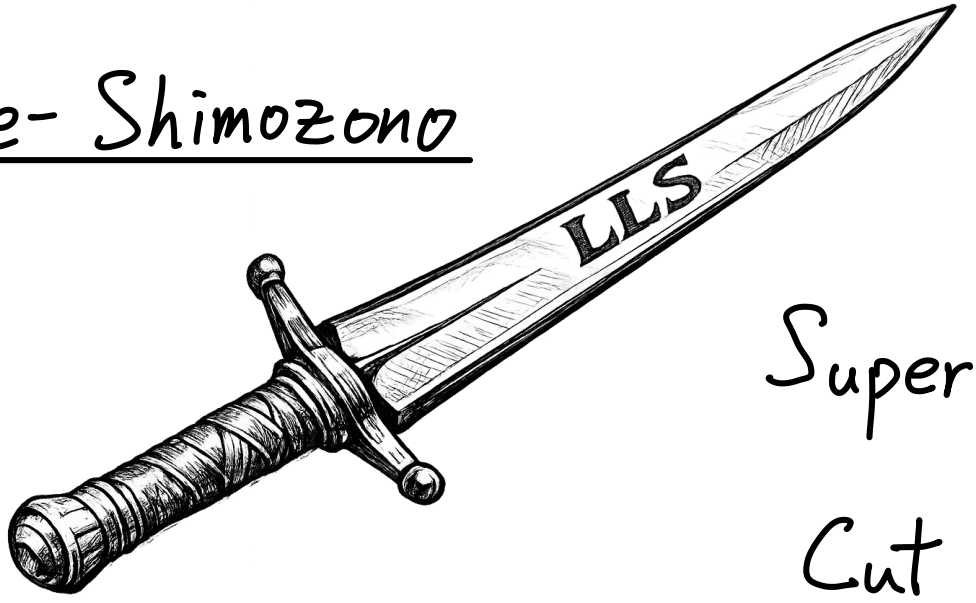
Goal: Find a combinatorial proof of this positivity.

How to compute $\tilde{J}^{w}_{\lambda}(y)$?



Sword of

Lam-Lee-Shimozono

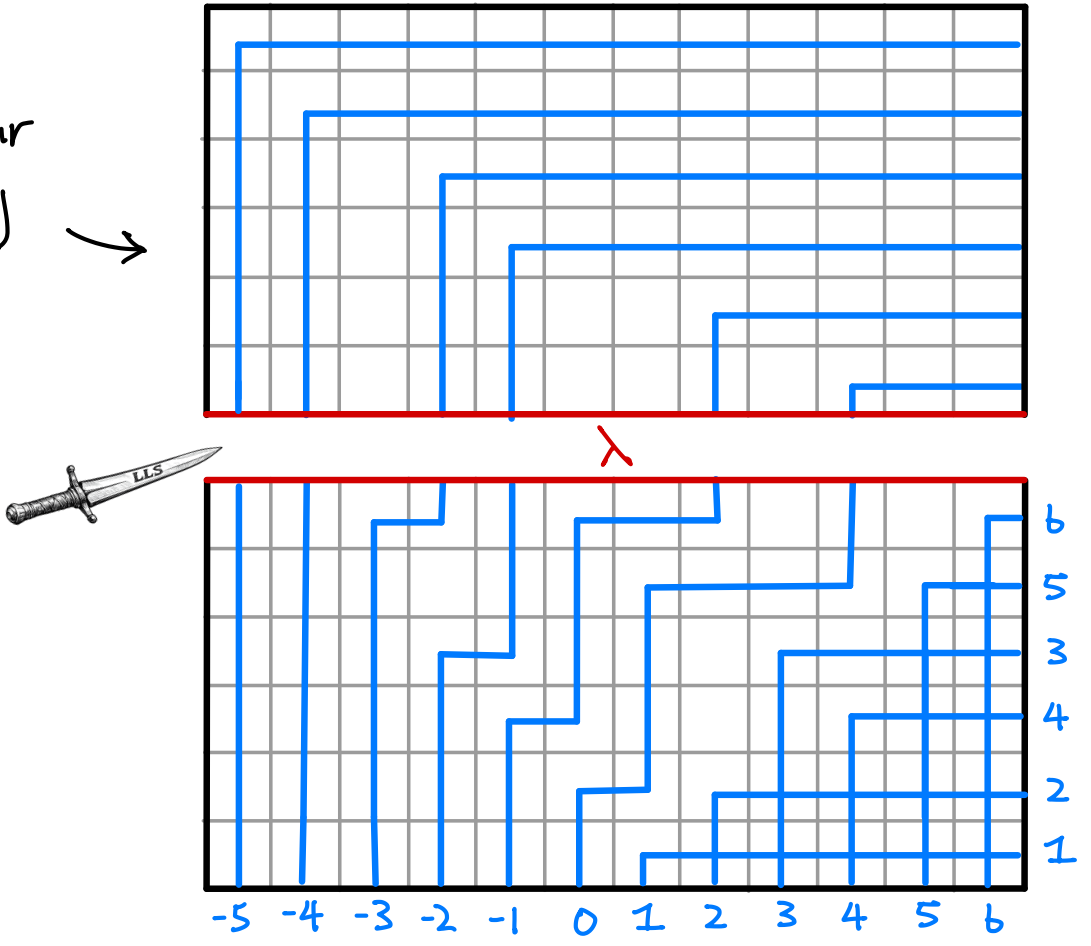


Super Power:

Cut BPDs

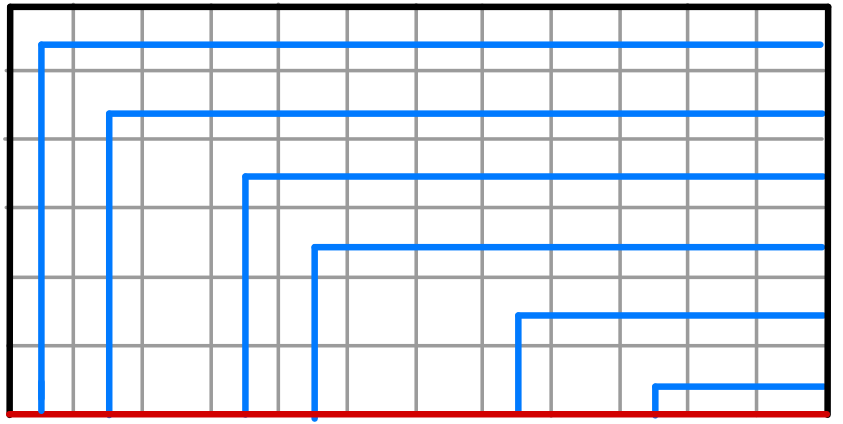
Double Schur

$S_\lambda(x;y)$ →



Double Schur

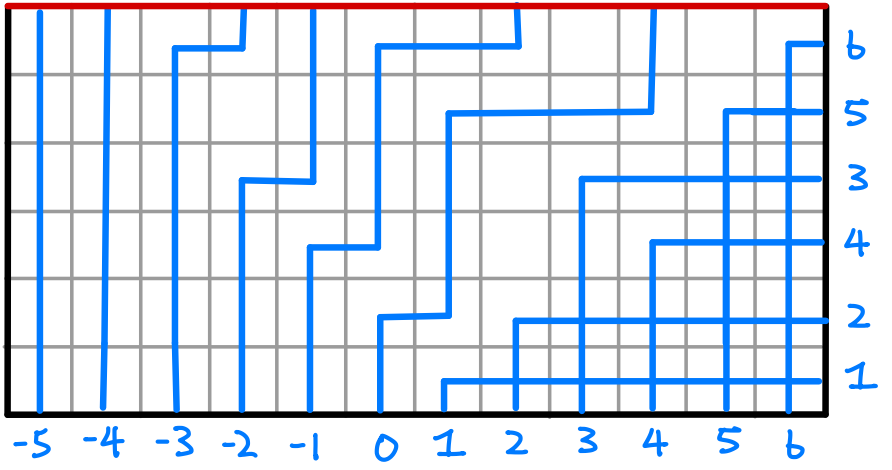
$S_\lambda(x;y)$ →



λ

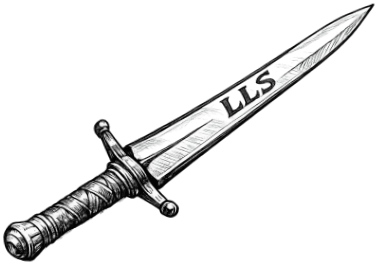
$\dot{J}^{w}_{\lambda}(y) =$

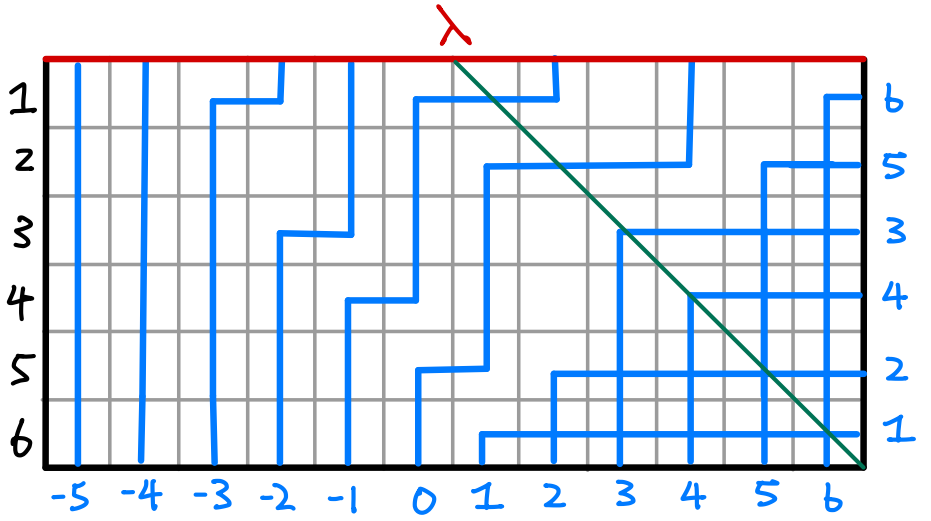




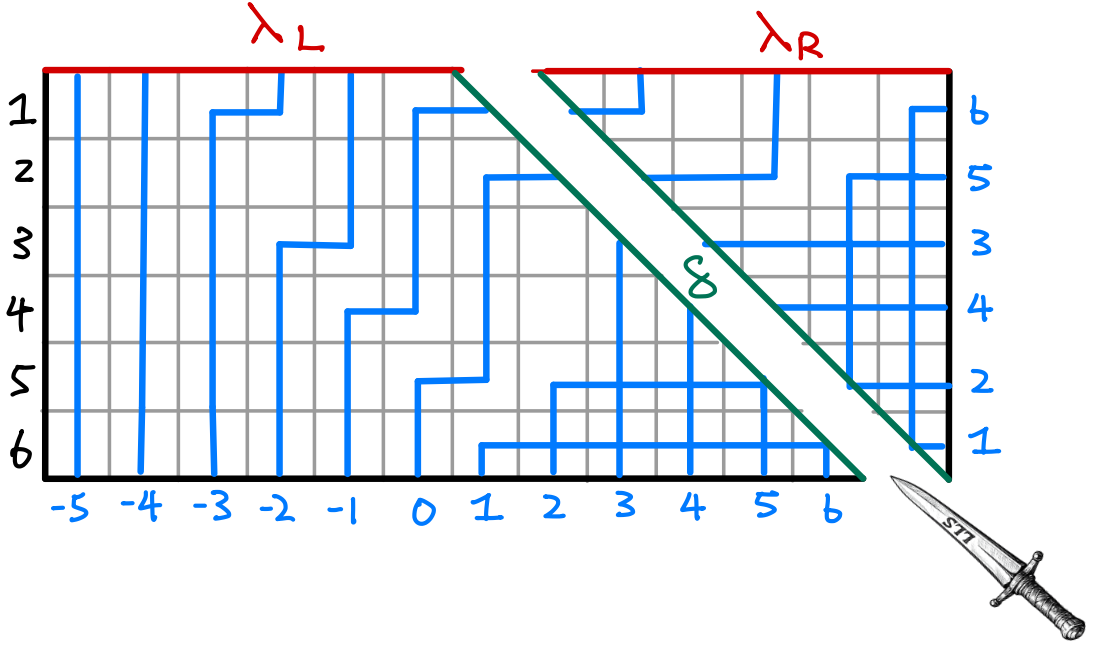
$$\sum \mathrm{wt}\left(\lambda \text{ rectangle } w\right)$$

Second cut: Diagonal cut





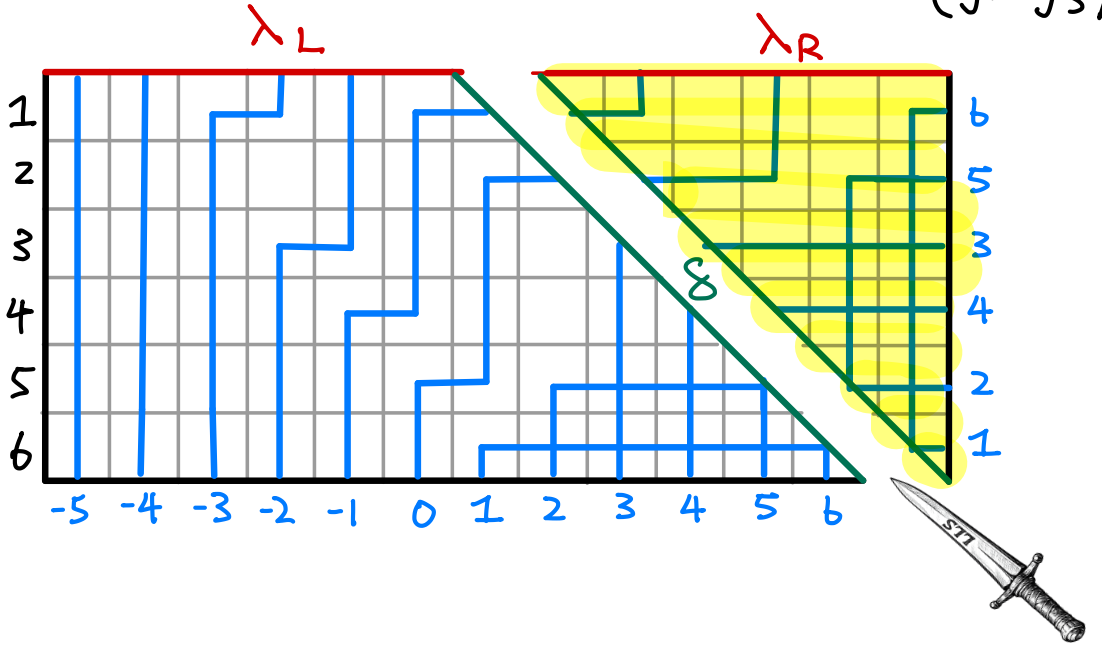
Second cut: Diagonal cut



$$J_\lambda^{-w}(y) = \sum_{\delta} \quad \lambda_L \;\; \delta \quad \times \quad \lambda_R \;\; \delta$$

Second cut: Diagonal cut

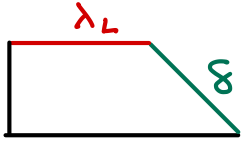
$(y_1 - y_3)(y_1 - y_5)$



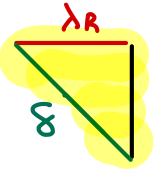
Positive!



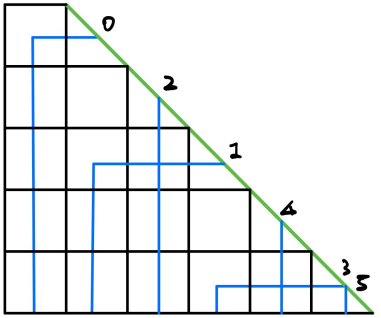
$$J_\lambda^{-w}(y) = \sum_{\delta}$$



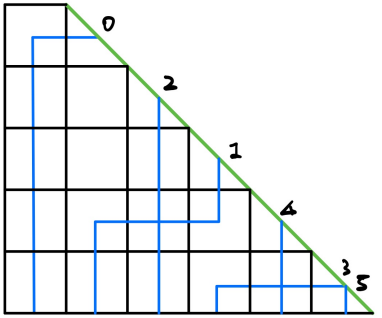
$\times$



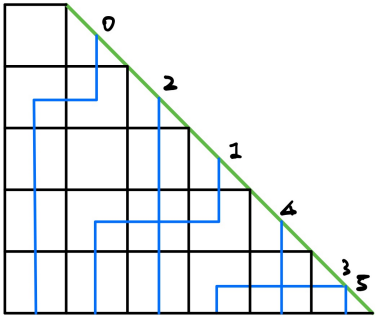
Trapezoid BPDs



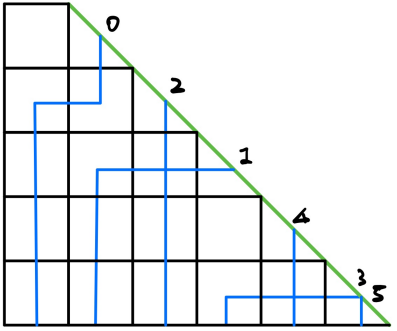
$(y_2 - y_1)(y_4 - y_3)$ +



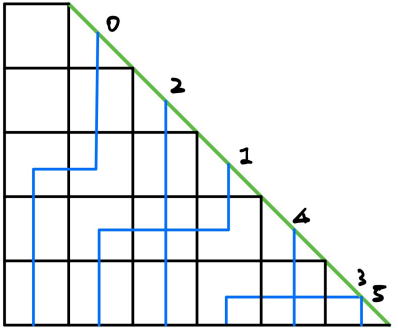
$(y_2 - y_1)(y_3 - y_1)$ +



$(y_1 - y_0)(y_3 - y_1)$

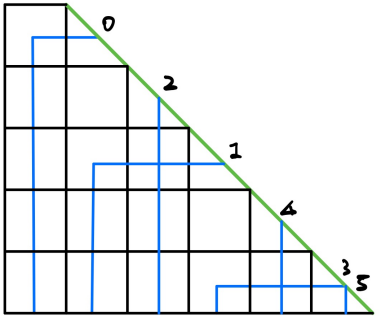


+ $(y_1 - y_0)(y_4 - y_3)$ +

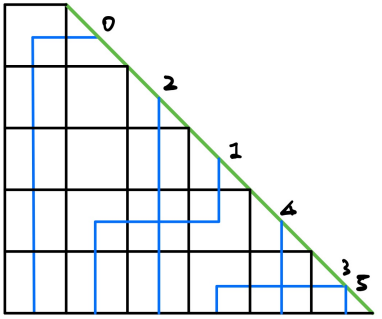


$(y_1 - y_0)(y_2 - y_0)$

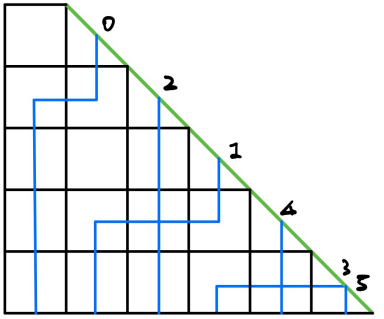
Trapezoid BPDs



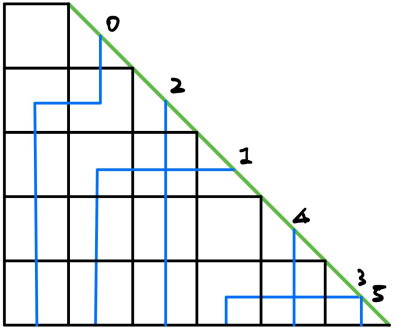
$(y_2 - y_1)(y_4 - y_3)$ +



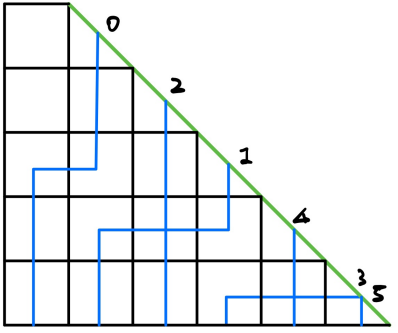
$(y_2 - y_1)(y_3 - y_1)$ +



$(y_1 - y_0)(y_3 - y_1)$



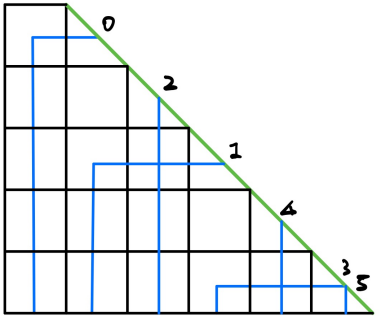
+ $(y_1 - y_0)(y_4 - y_3)$ +



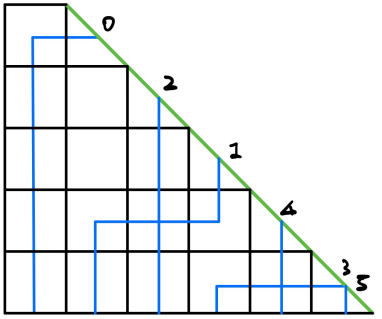
$(y_1 - y_0)(y_2 - y_0)$

Not Positive ?

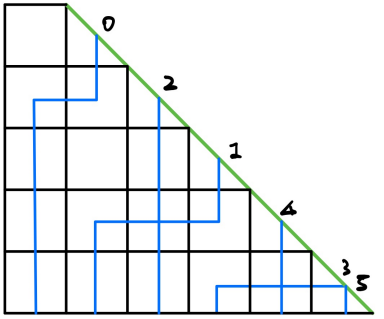
Trapezoid BPDs



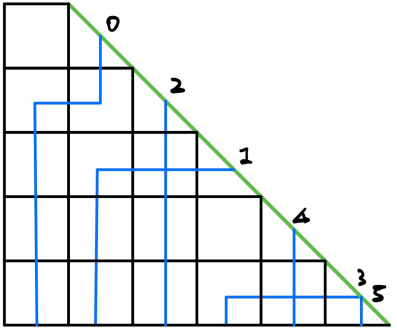
$(y_2 - y_1)(y_4 - y_3)$ +



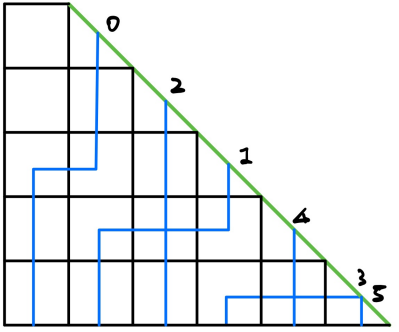
$(y_2 - y_1)(y_3 - y_1)$ +



$(y_1 - y_0)(y_3 - y_1)$



+ $(y_1 - y_0)(y_4 - y_3)$ +

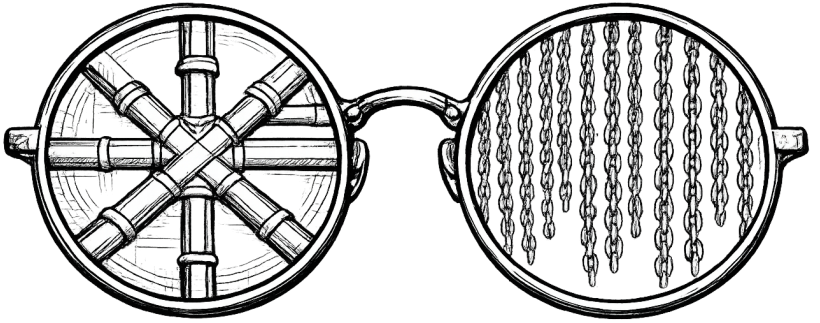


$(y_1 - y_0)(y_2 - y_0)$

$= (y_2 - y_0)(y_4 - y_0)$

Positive !!

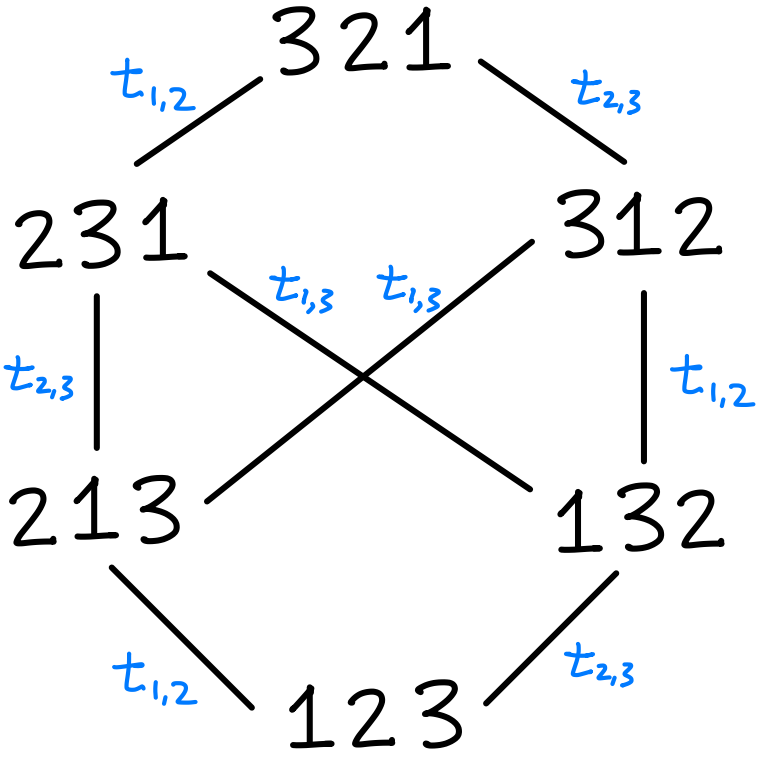
Glasses of Lenart and Sottile



Super Power: Pipes ⟶ Chains

Bruhat Order

- Partial order $\leq$ on S_n
- $u \lessdot u t_{a,b}$
  - $a<b$, $u(a)<u(b)$
  - Numbers in between are not in between
  - Coxeter length increases by 1



Increasing k-chain

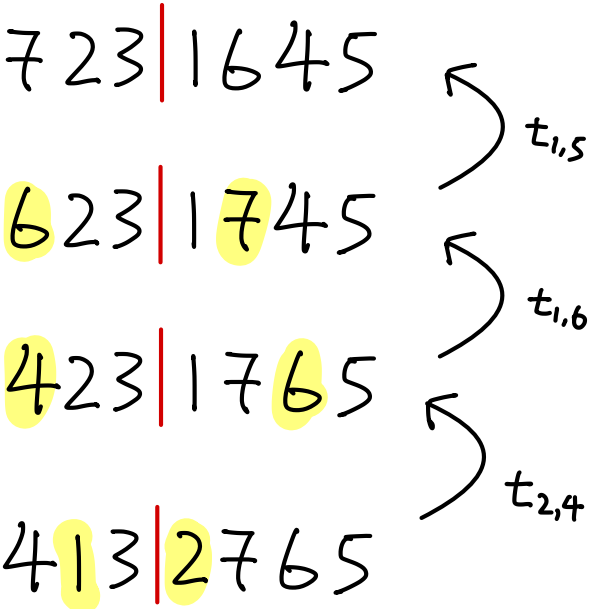
$u_1 \lessdot u_2 \lessdot u_3 \lessdot \cdots \lessdot u_m$

$u_{i+1} = u_i \, t_{a_i, b_i}$

Requirements:

- $a_i \leq k < b_i$
- Smaller number swapped is increasing

Ex. Inc. 3-chain



$4132765 \xrightarrow{3} 7231645$

Increasing k-chain

- $wt\left(u \xrightarrow[z]{k} w\right) = \prod_i (z - y_i)$ $i =$ fix points right of k

$$4132765 \xrightarrow{3} 6231745$$

$$623|1745$$
$$413|2765$$

$$wt\left(u \xrightarrow[x_3]{3} w\right) = (x_3 - y_7)(x_3 - y_5)$$

Increasing k-chain

- $wt\left(u \xrightarrow[z]{k} w\right) = \prod_i (z - y_i)$ $i =$ fix points right of k

$$4132765 \xrightarrow{3} 6231745 \xrightarrow{2} 7421635$$

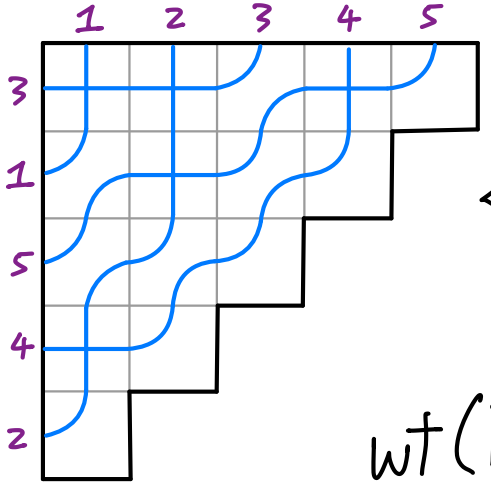
$$74|2|163|5$$
$$623|1|745$$
$$413|27|65$$

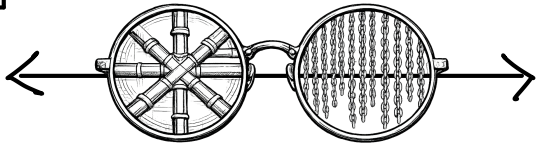
$$wt\left(u \xrightarrow[x_3]{3} \xrightarrow[x_2]{2} w\right)$$
$$= (x_3 - y_7)(x_3 - y_5)$$
$$(x_2 - y_1)(x_2 - y_5)$$

Increasing chain & PDs

Thm [Lenart-Sottile]

Pipe dreams of u $\Longleftrightarrow$ $u \xrightarrow{1} \xrightarrow{2} \cdots \xrightarrow{n-1} w_0$





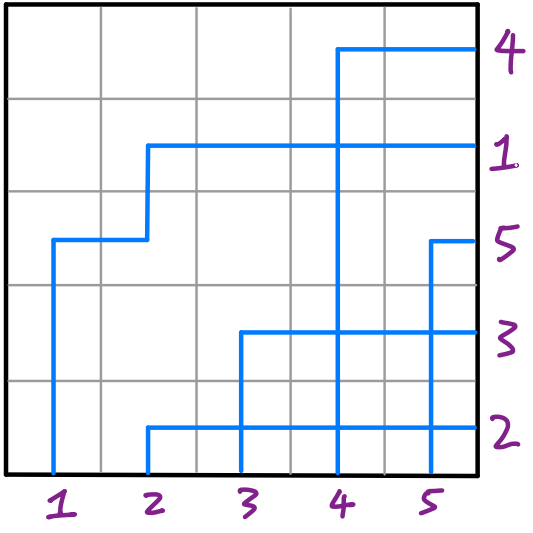
5	4	3	2	1
5	4	3	2	1
5	3	4	1	2
5	1	3	4	2
3	1	5	4	2

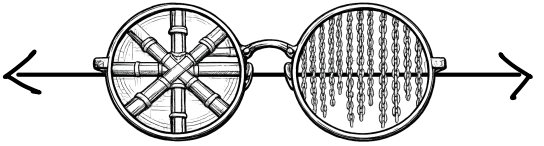
$$wt(D) = wt\left(u \xrightarrow[x_1]{1} \xrightarrow[x_2]{2} \cdots \xrightarrow[x_{n-1}]{n-1} w_0\right)$$

Increasing chain & BPDs

Thm [Yu]

Bumpless Pipe dreams of $u \longleftrightarrow u \xrightarrow{n-1} \xrightarrow{n-2} \cdots \xrightarrow{1} w_0$

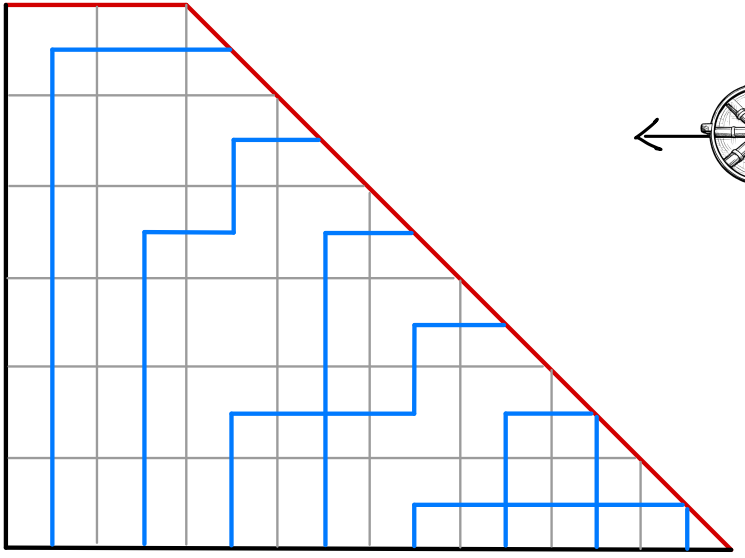


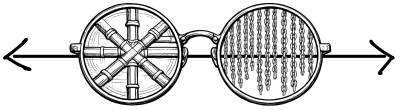


5 | 4 3 2 1
4 | 5 | 3 2 1
4 2 | 5 | 3 1
4 1 5 | 3 | 2
4 1 5 3 | 2

$$wt(D) = wt\left(u \xrightarrow[x_{n-1}]{n-1} \xrightarrow[x_{n-2}]{n-2} \cdots \xrightarrow[x_1]{1} w_0\right)$$

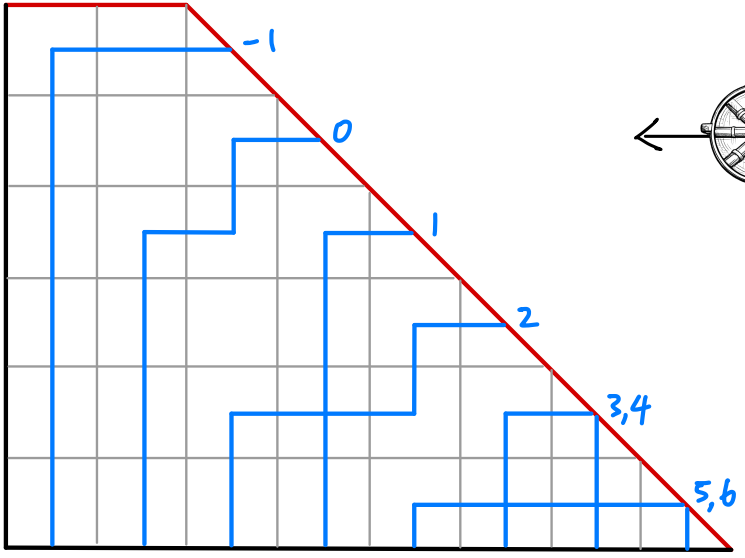
Increasing chain and Trapezoid BPDs

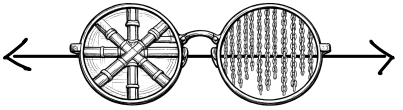




1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

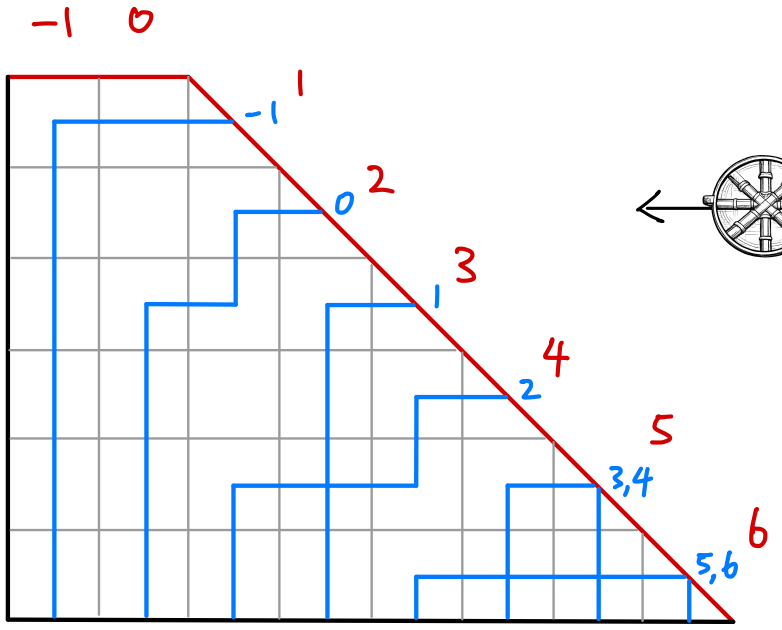
Increasing chain and Trapezoid BPDs

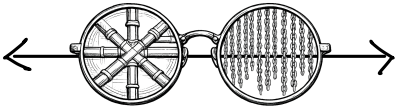




1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
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-1	0	2	1	5	4	6	3

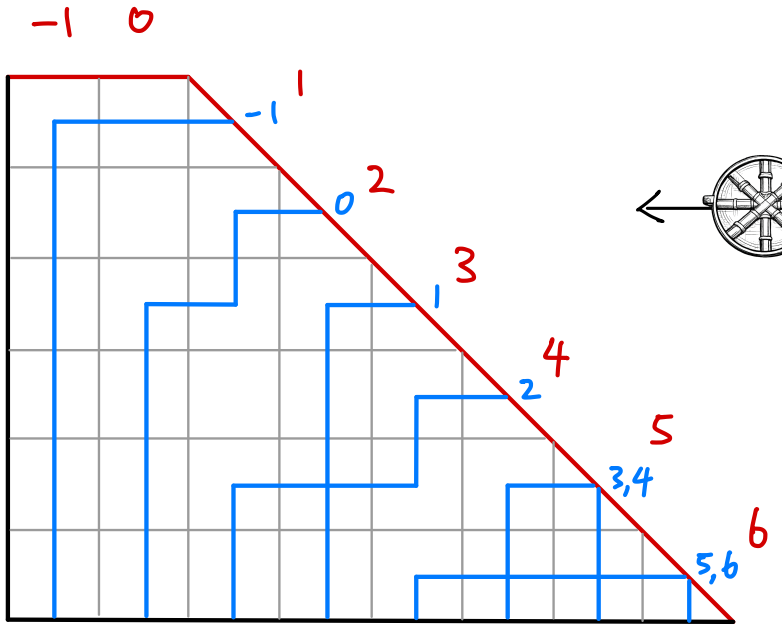
Increasing chain and Trapezoid BPDs

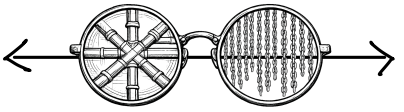




1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
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-1	0	2	1	5	4	6	3

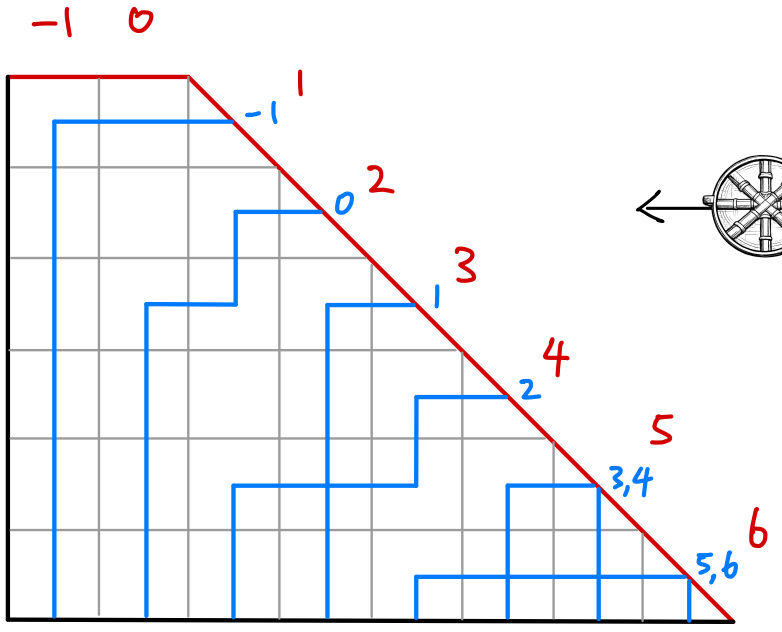
Increasing chain and Trapezoid BPDs

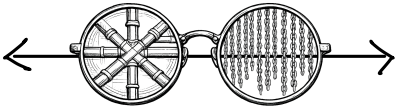




-1	0	1	2	3	4	5	6
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

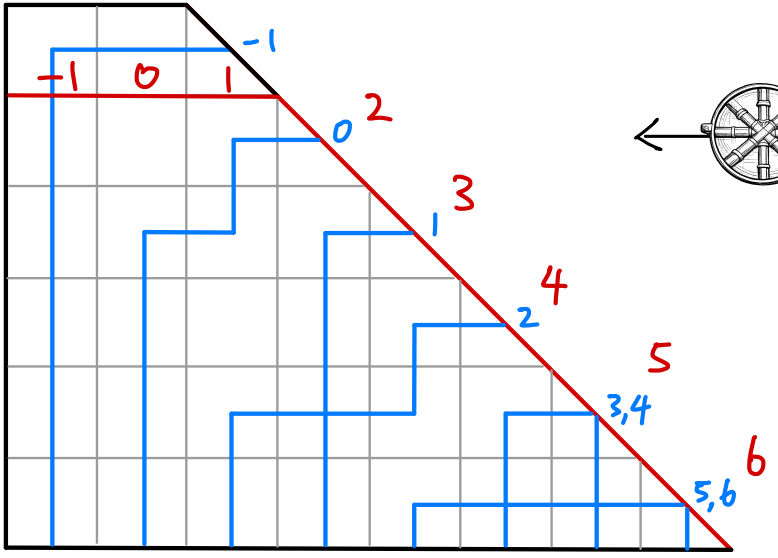
Increasing chain and Trapezoid BPDs

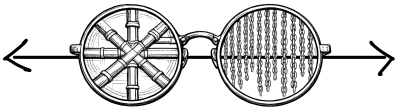




-1	0	1	2	3	4	5	6
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

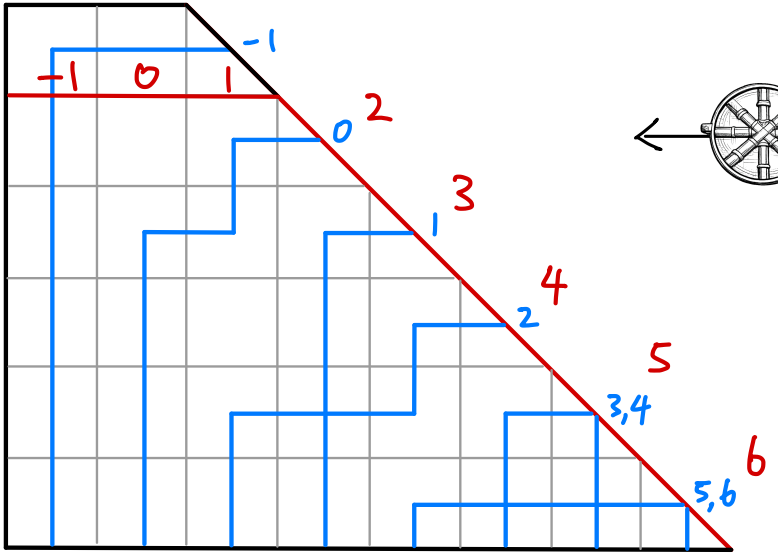
Increasing chain and Trapezoid BPDs

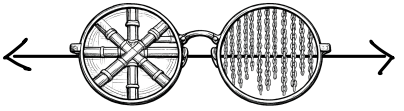




-1	0	1	2	3	4	5	6
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

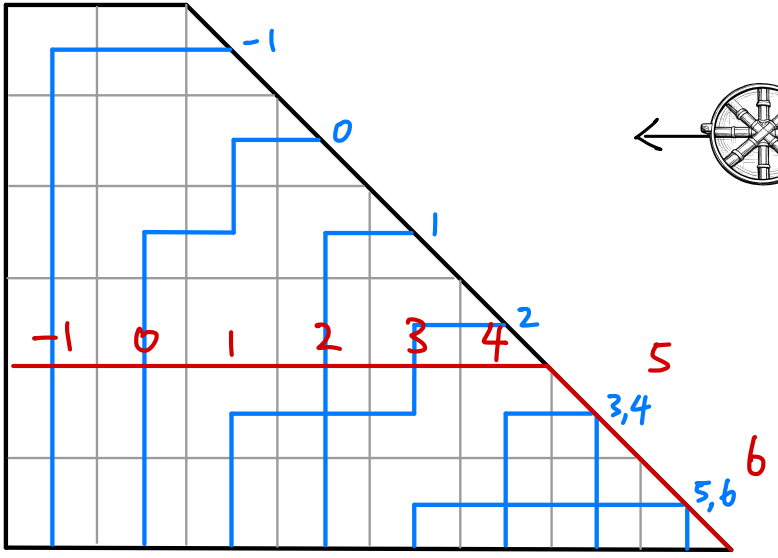
Increasing chain and Trapezoid BPDs

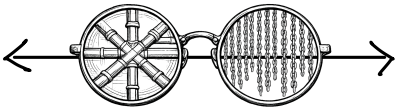




-1	0	1	2	3	4	5	6
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

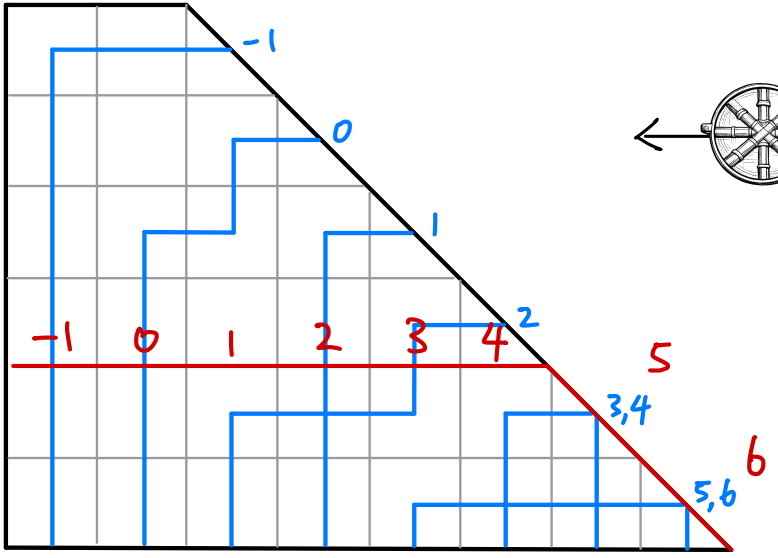
Increasing chain and Trapezoid BPDs

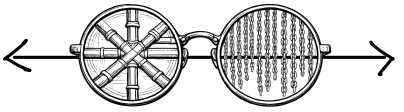




-1	0	1	2	3	4	5	6
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

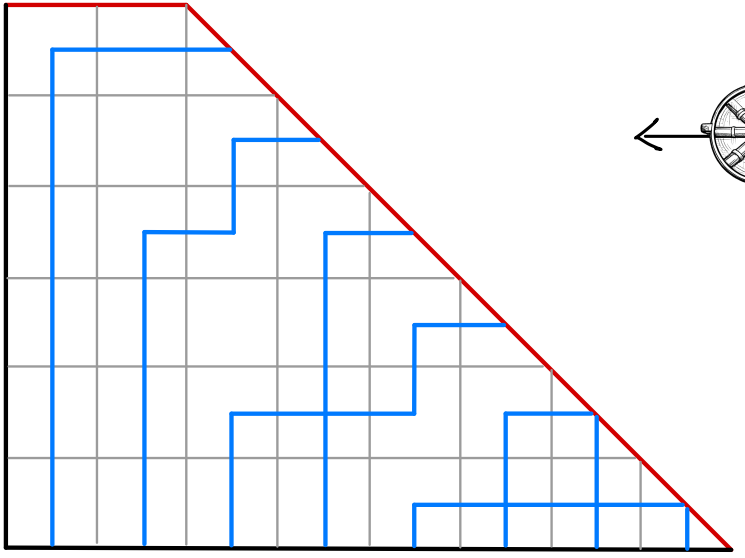
Increasing chain and Trapezoid BPDs

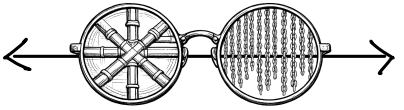




-1	0	1	2	3	4	5	6
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
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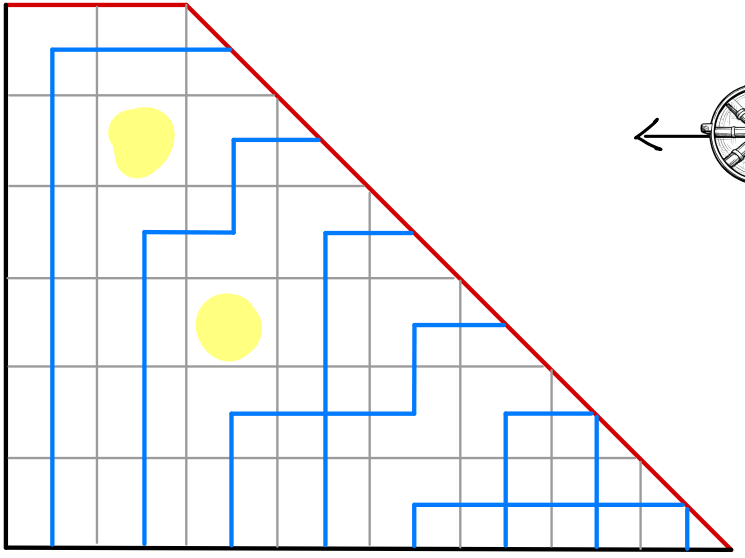
Increasing chain and Trapezoid BPDs

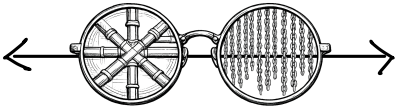




1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

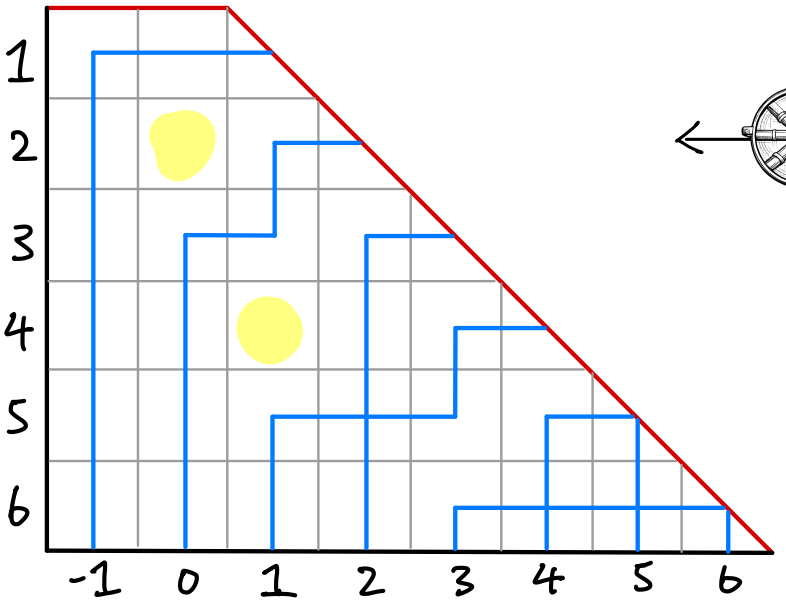
Increasing chain and Trapezoid BPDs

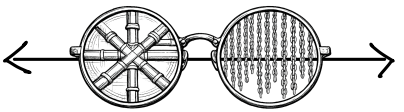




1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

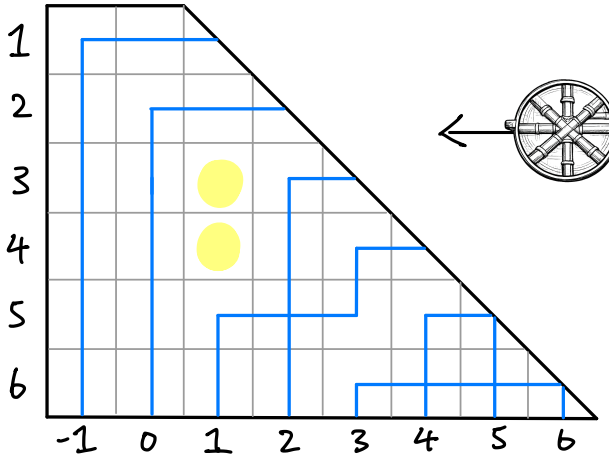
Increasing chain and Trapezoid BPDs

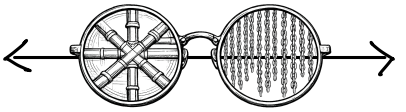




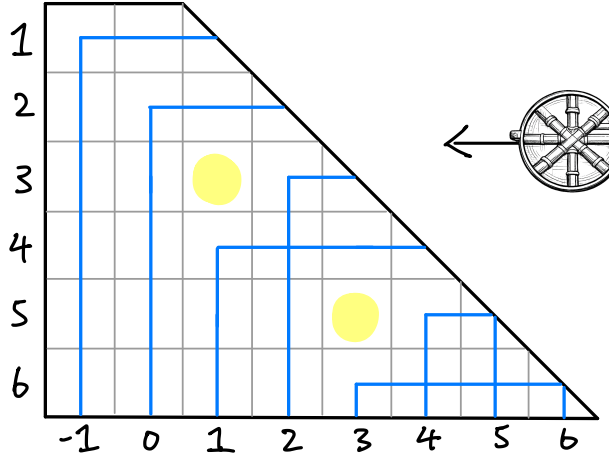
1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	1	3	4	5	2	6	0
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

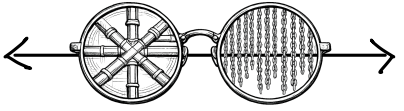
$$wt(D) = wt\left(c : u \xrightarrow[y_1]{\alpha_1} \cdots \xrightarrow[y_m]{\alpha_m} w\right) \Big/ \prod_{i<j} (y_i - y_j)$$



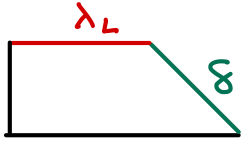
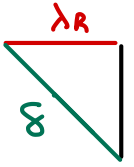


1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	0	3	4	5	2	6	1
-1	0	2	4	5	3	6	1
-1	0	2	3	5	4	6	1
-1	0	2	1	5	4	6	3

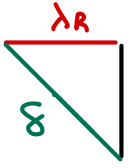


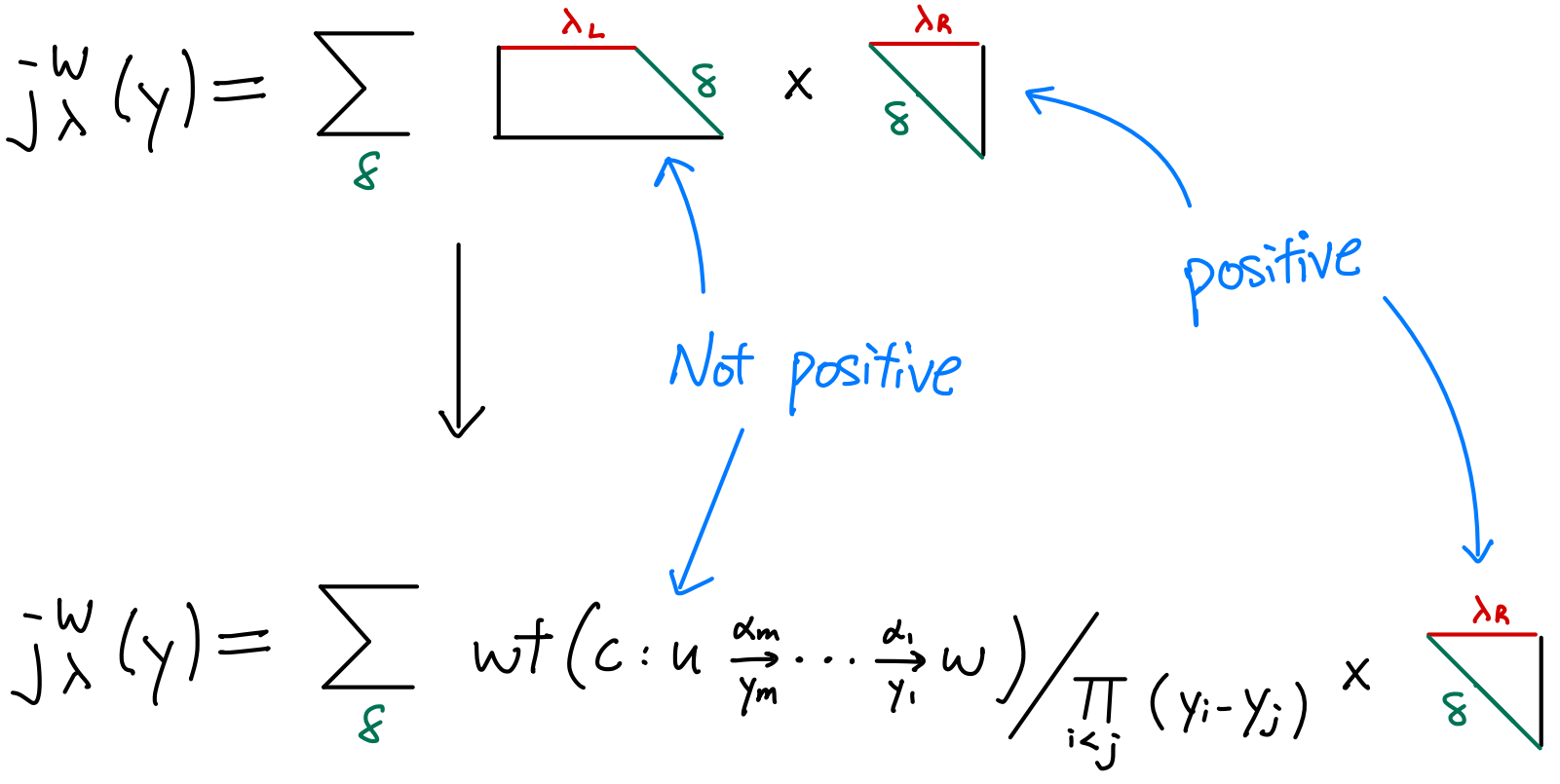


1	2	3	4	5	0	6	-1
-1	2	3	4	5	1	6	0
-1	0	3	4	5	2	6	1
-1	0	2	4	5	3	6	1
-1	0	2	1	5	4	6	3
-1	0	2	1	5	4	6	3

$$J^{\vec{w}}_{\lambda}(y) = \sum_{\delta}$$  $\times$ 



$$J^{\vec{w}}_{\lambda}(y) = \sum_{\delta} wt(c : u \to w) \Big/ \prod_{i<j} (y_i - y_j) \times$$ 


$J^{\vec{w}}_{\lambda}(y) = \sum_{\xi}$
λ_L
ξ
$\times$
λ_R
ξ
positive
Not positive
$J^{\vec{w}}_{\lambda}(y) = \sum_{\xi} wt\left(c : u \xrightarrow[y_m]{\alpha_m} \cdots \xrightarrow[y_1]{\alpha_1} w\right) \Big/ \prod_{i<j}(y_i - y_j) \times$
λ_R
ξ

Gauntlet of Samuel



Super Power:

Swap labels in chains

Swapping Labels



Prop. [Samuel] Fix u, w

$$\sum_{u \xrightarrow{a} \xrightarrow{b} w} wt\left(u \xrightarrow[z_a]{a} \xrightarrow[z_b]{b} w\right) = \sum_{u \xrightarrow{b} \xrightarrow{a} w} wt\left(u \xrightarrow[z_b]{b} \xrightarrow[z_a]{a} w\right)$$

Thus,

$$\sum_{u \xrightarrow{\alpha_1} \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_m} w} wt\left(u \xrightarrow[z_1]{\alpha_1} \xrightarrow[z_2]{\alpha_2} \cdots \xrightarrow[z_m]{\alpha_m} w\right)$$

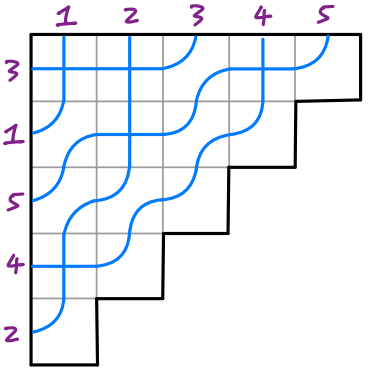
is invariant under swapping labels of the chains.

Example: Schubert polynomials

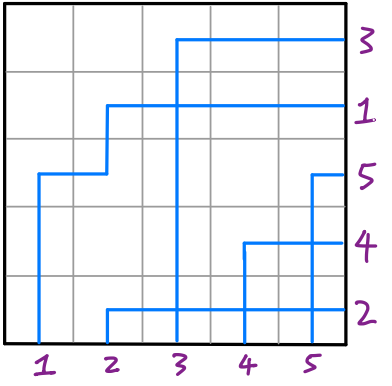
$$\sum_{u \xrightarrow{1} \xrightarrow{2} \xrightarrow{3} \xrightarrow{4} w_0} wt\left(u \xrightarrow[x_1]{1} \xrightarrow[x_2]{2} \xrightarrow[x_3]{3} \xrightarrow[x_4]{4} w_0\right)$$



$$\sum_{u \xrightarrow{4} \xrightarrow{3} \xrightarrow{2} \xrightarrow{1} w_0} wt\left(u \xrightarrow[x_4]{4} \xrightarrow[x_3]{3} \xrightarrow[x_2]{2} \xrightarrow[x_1]{1} w_0\right)$$



$$\mathfrak{S}_u(x;y)$$



Prop. [Samuel] Fix u, w



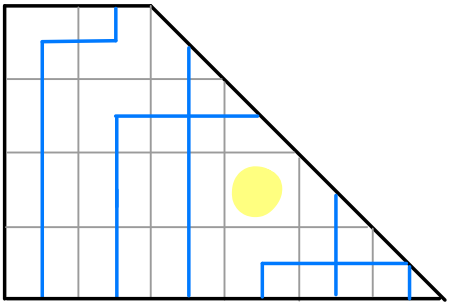
$$\sum_{u \xrightarrow{a} \xrightarrow{b} w} wt\left(u \xrightarrow[z_1]{a} \xrightarrow[z_2]{b} w\right) = \sum_{u \xrightarrow{b} \xrightarrow{a} w} wt\left(u \xrightarrow[z_2]{b} \xrightarrow[z_1]{a} w\right)$$

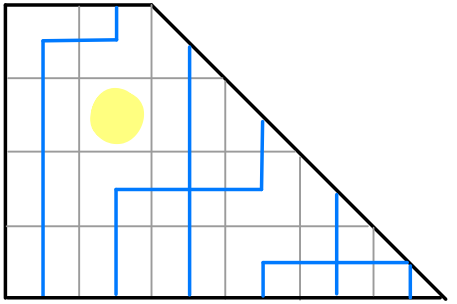
Thm. [Samuel]

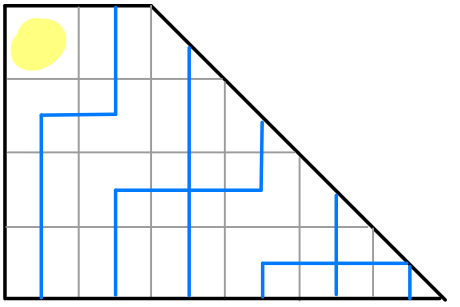
$$G_{ww_0}(x_1 - z) \cdots (x_k - z) = \sum_{u \to w} G_{uw_0}\, wt\left(u \xrightarrow[z]{k} w\right)$$

$$LHS = [G_{uw_0}]G_{ww_0}\,(x_1 - z_1)\cdots(x_a - z_1)(x_1 - z_2)\cdots(x_b - z_2)$$

$$RHS = [G_{uw_0}]G_{ww_0}\,(x_1 - z_2)\cdots(x_b - z_2)(x_1 - z_1)\cdots(x_a - z_1)$$







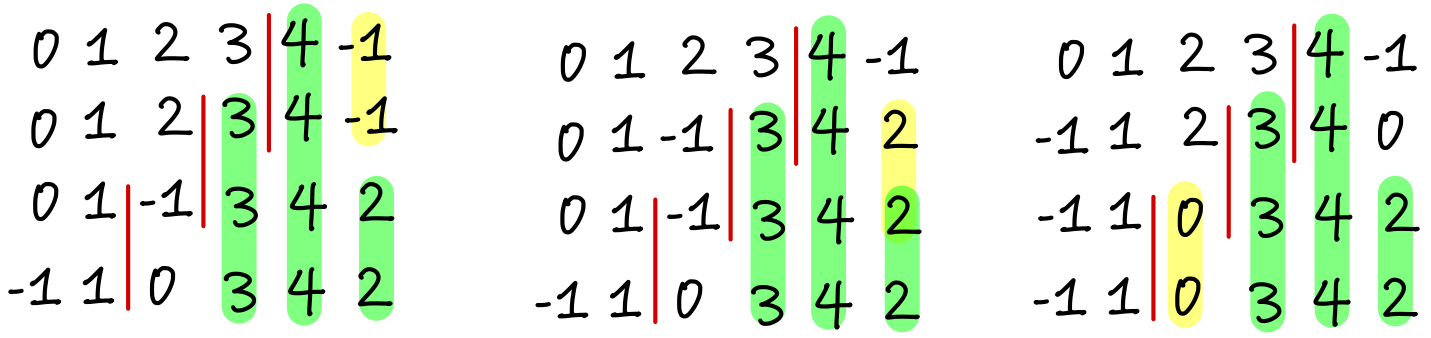
Not Positive →

$$(y_3 - y_2) + (y_2 - y_0) + (y_1 - y_{-1})$$

0	1	2	3	4	-1
-1	1	2	3	4	0
-1	1	0	3	4	2
-1	1	0	3	4	2

0	1	2	3	4	-1
-1	1	2	3	4	0
-1	1	2	3	4	0
-1	1	0	3	4	2

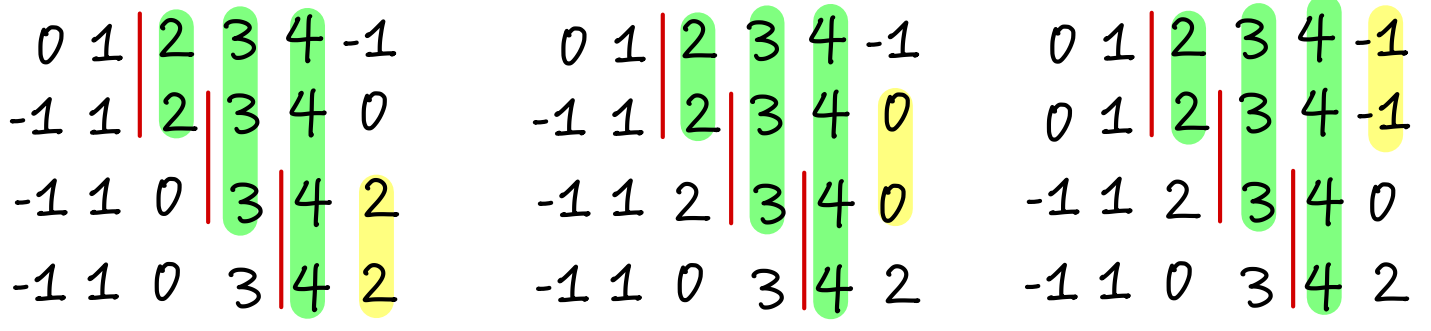
0	1	2	3	4	-1
0	1	2	3	4	-1
-1	1	2	3	4	0
-1	1	0	3	4	2

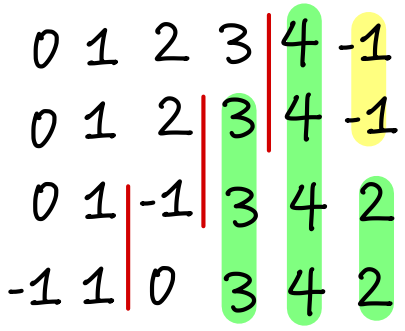


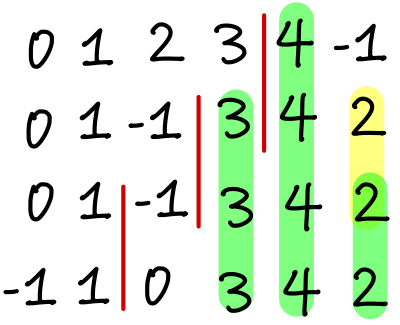
$$(Y_3 - Y_{-1}) + (Y_2 - Y_2) + (Y_1 - Y_0)$$

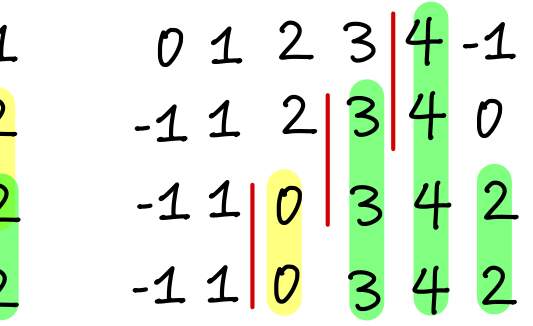
Not Positive →

$$(Y_3 - Y_2) + (Y_2 - Y_0) + (Y_1 - Y_{-1})$$







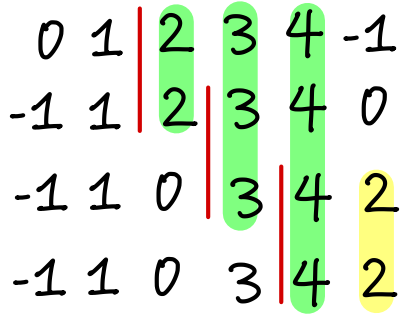


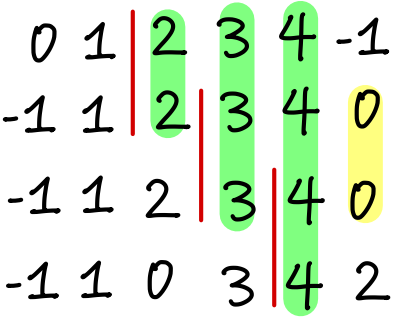
All positive or 0 →

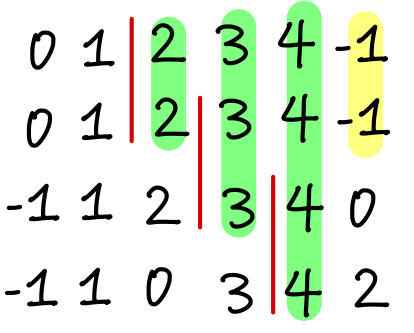
$$(Y_3 - Y_{-1}) + (Y_2 - Y_2) + (Y_1 - Y_0)$$

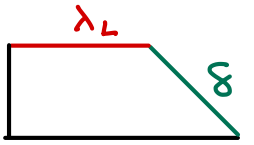
Not Positive →

$$(Y_3 - Y_2) + (Y_2 - Y_0) + (Y_1 - Y_{-1})$$







Prop (C.-Yu) For u, w coming from 

$$wt\left(u \xrightarrow[y_1]{\alpha_1} \cdots \xrightarrow[y_m]{\alpha_m} w\right) \Big/ \prod_{i<j} (y_i - y_j)$$

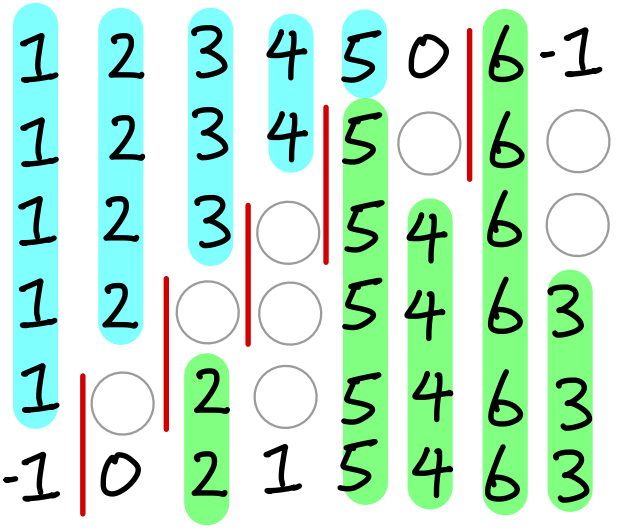
is either zero or

$$\prod (y_{>0} - y_{\leq 0}) .$$

In particular, it is

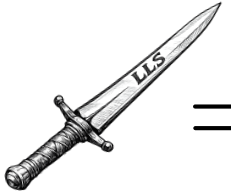
positive.

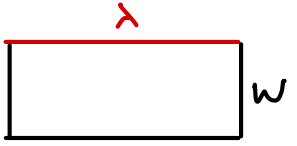
Proof:

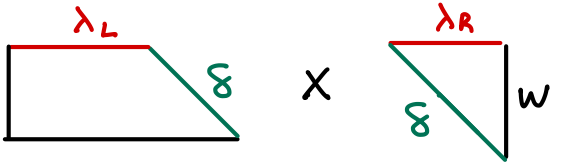


Summary

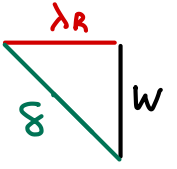
Double Edelman-Greene coefficients $J_{\lambda}^{-w}(y)$

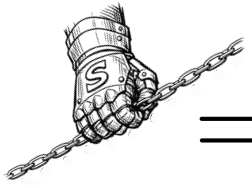


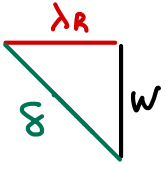
$=$ 

$=$  $\sum_{\delta}$ 



$= \sum_{\delta} wt\left(u \xrightarrow[y_m]{\alpha_m} \cdots \xrightarrow[y_1]{\alpha_1} w\right) \Big/ \prod_{i<j}(y_i - y_j)$ 



$= \sum_{\delta} wt\left(u \xrightarrow[y_1]{\alpha_1} \cdots \xrightarrow[y_m]{\alpha_m} w\right) \Big/ \prod_{i<j}(y_i - y_j)$ 

Each term is positive !!!

Final Remarks

$$\overset{-w}{J}_{\lambda}(y) = \sum_{\delta} wt\left(u \xrightarrow[y_1]{\alpha_1} \cdots \xrightarrow[y_m]{\alpha_m} w\right) \Big/ \prod_{i<j}(y_i - y_j) \times \begin{array}{c} \lambda_R \\ \delta \quad w \end{array}$$

Thm: For $w \in S_n$, $\overset{-w}{J}_{\lambda}(y) \in \mathbb{Z}_{\geq 0}[y_i - y_j : i \lessdot j]$

Moreover, in each term:

degree of $(y_{>0} - y_{>0}) \leq 1$ $\quad \begin{array}{c} \lambda_R \\ \delta \quad w \end{array}$

degree of $(y_{>0} - y_{\leq 0}) \leq 1$ $\quad \sum_{\delta} wt\left(u \xrightarrow[y_1]{\alpha_1} \cdots \xrightarrow[y_m]{\alpha_m} w\right) \Big/ \prod_{i<j}(y_i - y_j)$

degree of $(y_{\leq 0} - y_{\leq 0}) = 0$

Final Remarks

Our formula also works for $w \in S_{\mathbb{Z}}$,

$$J_\lambda^{-w}(y) = \sum \omega\left(J_{\lambda_1}^{-w_1}(y)\right) J_{\lambda_2}^{-w_2}(y)$$

ω preserves positivity

Thm [Anderson] $J_\lambda^{-w}(y) \in \mathbb{Z}_{\geq 0}[y_i - y_j : i \lessdot j]$

Moreover, in each term:

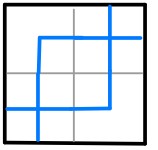
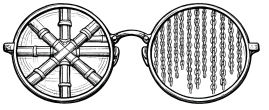
degree of $(y_{>0} - y_{>0}) \leq 1$

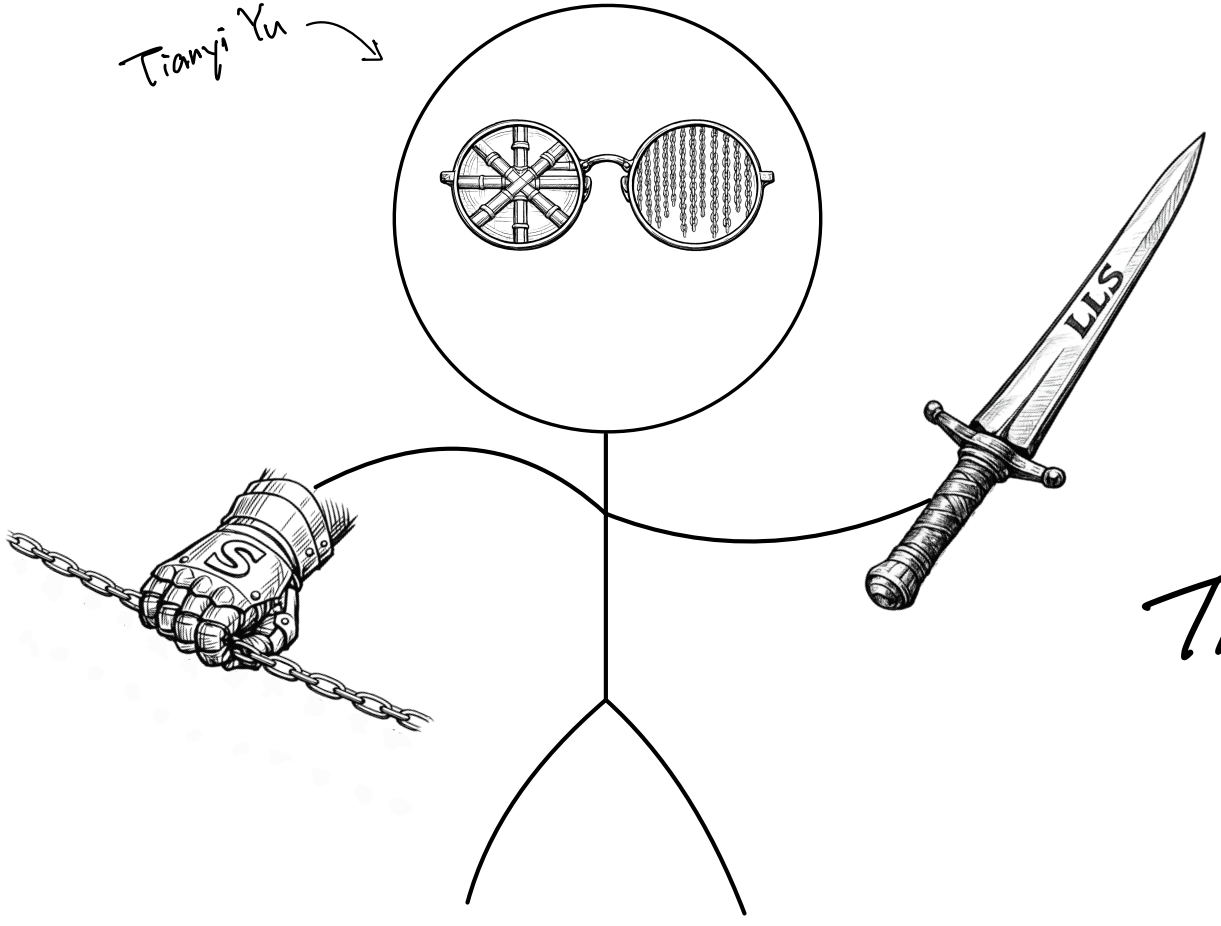
degree of $(y_{>0} - y_{\leq 0}) \leq 2$

degree of $(y_{\leq 0} - y_{\leq 0}) \leq 1$

We can also show this fact combinatorially.

Final Remarks: What's next?

- K-Theory version (allow )
- Triple EG coefficients $j_{\lambda}^{w}(y;z)$
- Positivity via  + 
- Schubert structure constants
- Etc...



Thank you !!