A positive combinatorial formula for the double Edelman-Greene coefficients

Jack Chou

joint with Tianyi Yu

4.1.2026

What is Algebraic Combinatorics ?

Motivation: Algebraic Geometry (Cohomology, Varieties)

Rep. Theory (Characters)

Combinatorics: Posets, Permutations, Partitions

Diagrams, Tableaux, Crystals, ...

Hilbert's 15th Problem: Schubert Calculus

Before: Enumerative Geometry

How many lines intersect 4 generic lines in $\mathbb{R}^3$?

Answer: 2

Hilbert's 15th Problem: Schubert Calculus

Now: Flag Variety

$$Fl_n(\mathbb{C}) = \{ \phi = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n : \dim(V_i) = i \}$$

$$\cong GL_n(\mathbb{C})/B \qquad B = \text{Upper Triangular Matrices}$$

$$\cong \bigsqcup_{w \in S_n} BwB/B =: X_w^\circ \leftarrow \text{Schubert cells}$$

$[\overline{X_w^\circ}] \in H^*(Fl_n)$ has representatives called

Schubert variety X_w Schubert Polynomials $\mathfrak{S}_w$.

(Double) Schubert Polynomials

Represent classes in T-equivarient cohomology of flag varieties.

$$\mathfrak{S}_{2143} = (x_1-y_1)(x_3-y_3) + (x_1-y_1)(x_1-y_2) + (x_1-y_1)(x_2-y_1)$$

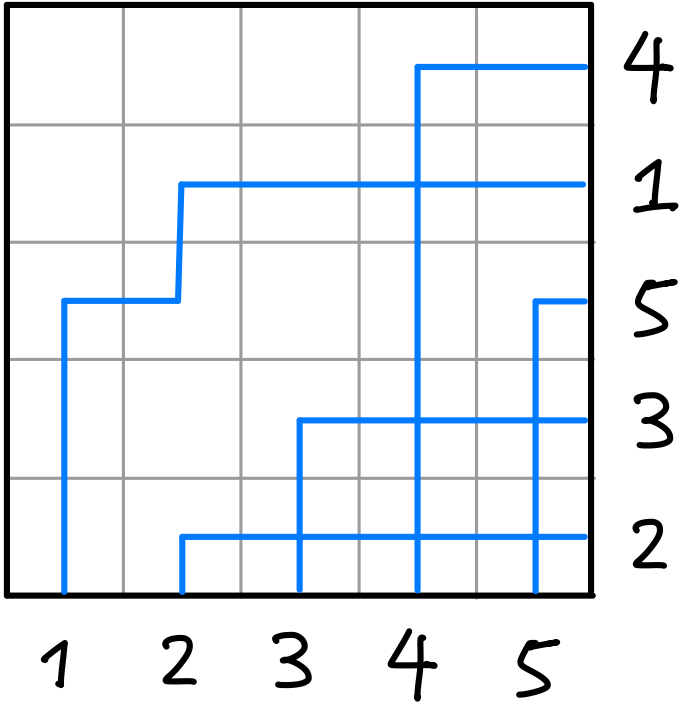
2143: permutation

$(x_1-y_1)(x_1-y_2)$: product of (x_i-y_j)

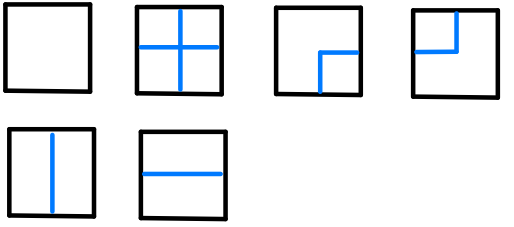
Can be defined recursively (divided difference)

Have many combinatorial formulae

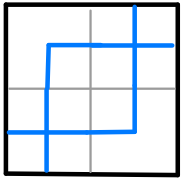
Bumpless Pipe Dreams: [Lam-Lee-Shimozono]



Tiles



No double crossings



$$\mathfrak{S}_w = \sum_{D \in \mathrm{BPD}(w)} \prod_{(i,j)=\square} (x_i - y_j)$$

Classical Schubert Calculus

FL_n Flag variety

Equivariant Cohomology:

Double Schubert Polynomials

Combinatorial Objects:

Pipe Dreams

Bumpless Pipe Dreams

Back-stable Schubert Calculus

FL_∞ Infinite Flag variety

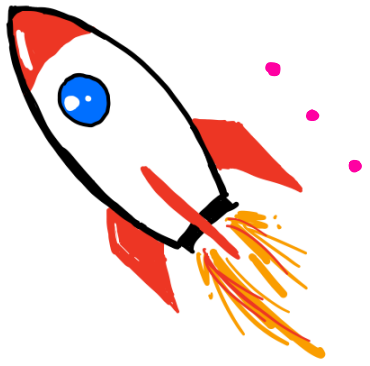
Back-stable

Double Schubert Functions

Infinite

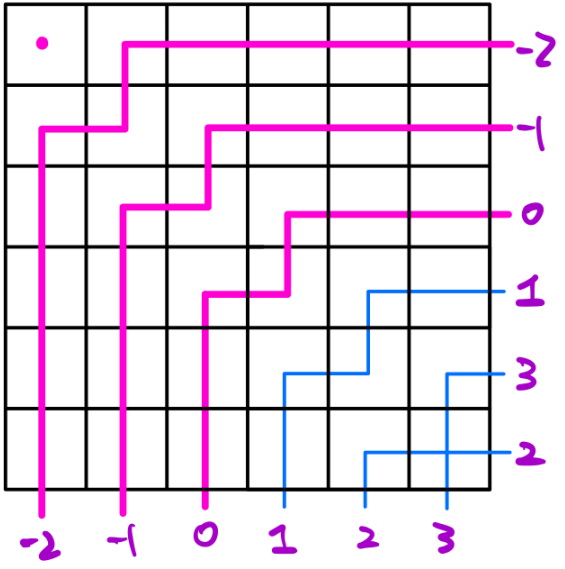
Bumpless Pipe Dreams

Backstable $\subset_w$



To Infinity and Beyond!

$$\overleftarrow{\subset}_{132} = (x_2 - y_2) + (x_1 - y_1) + (x_0 - y_0) + (x_{-1} - y_{-1}) + (x_{-2} - y_{-2}) + \cdots$$



Double EG Coeff:

$$\overleftarrow{G}_w = \sum_\lambda a_\lambda^w(x;y)\, s_\lambda(x;y)$$

$a_\lambda^w(x;y)$: Coefficients in $\mathbb{Z}[x;y]$

$s_\lambda(x;y)$: Double Schur

Double Edelman-Greene coefficients:

$$\overleftarrow{J}_\lambda^w(y) := a_\lambda^w(y;y) \qquad \text{Set } x_i \mapsto y_i$$

- Lam, Lee, and Shimozono defined the double Edelman-Greene coefficients $\bar{J}^{w}_{\lambda}(y)$.
- They showed that $\bar{J}^{w}_{\lambda}(y)$ are "positive" geometrically, and asked for combinatorial proof.
- We solve this problem using bumpless pipe dreams and increasing chains in Bruhat order.

Double EG Coeff:

Theorem [Lam-Lee-Shimozono]

$$j_\lambda^w(y) \in \mathbb{Z}_{\geq 0}[\, y_i - y_j : i \prec j \,]$$

$$1 \prec 2 \prec 3 \prec \cdots \prec -2 \prec -1 \prec 0$$

Positive

Ex. $2(y_1 - y_4)^2 (y_3 - y_{-1}) + (y_{-1} - y_0)$

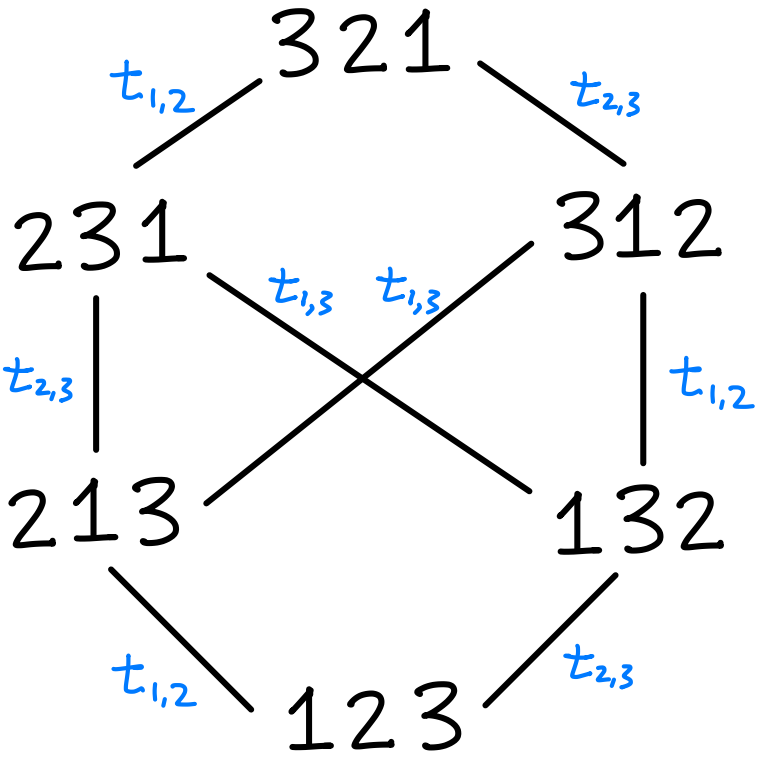
Goal: Find a combinatorial proof of this positivity.

Bruhat Order

- Partial order $\leq$ on S_n

  Containment of X_w

- $u \lessdot u t_{a,b}$
  - $a<b$, $u(a)<u(b)$
  - Numbers in between are not in between



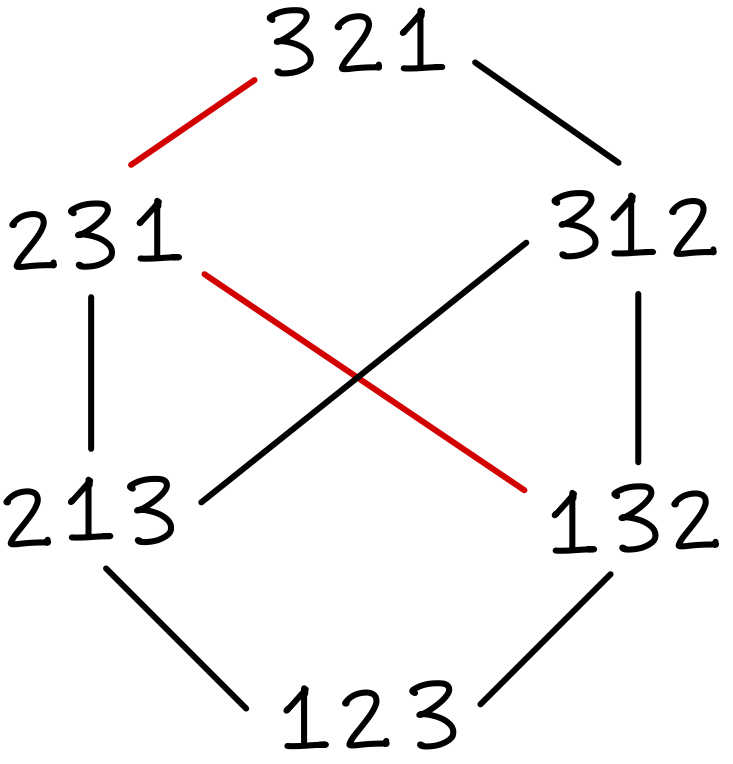
Increasing Chains

composition

For some label α,

$wt(132 \xrightarrow{\alpha} 321)$

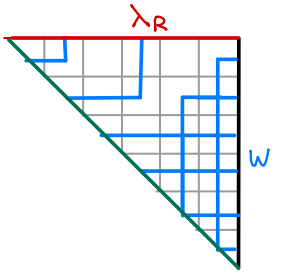
is a polynomial.



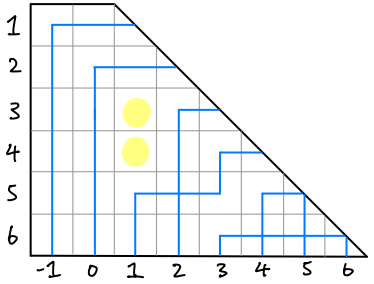
Thm: [C.-Yu]

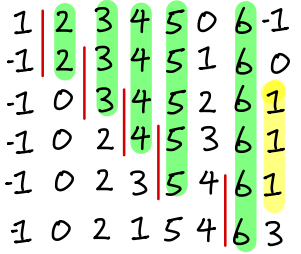
Double Edelman-Greene coefficients $J_{\lambda}^{w}(y)$

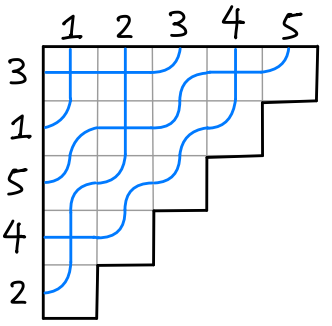
$$= \sum_{\mathcal{E}} wt\left(u \xrightarrow[y_1]{\alpha_1} \cdots \xrightarrow[y_m]{\alpha_m} v\right) \Big/ \prod_{i<j}(y_i - y_j) \times$$



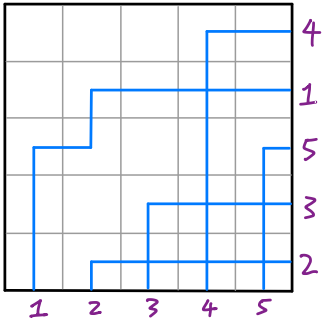
Each term is positive !!!

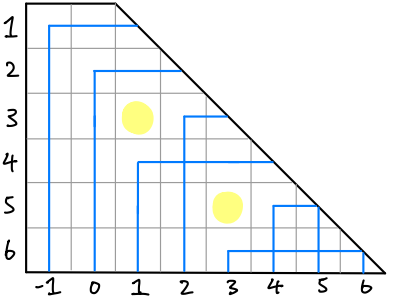

1 2 3 4 5 6
-1 0 1 2 3 4 5 6

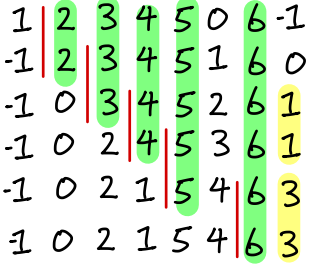

1 2 3 4 5 0 6 -1
-1 2 3 4 5 1 6 0
-1 0 3 4 5 2 6 1
-1 0 2 4 5 3 6 1
-1 0 2 3 5 4 6 1
-1 0 2 1 5 4 6 3


1 2 3 4 5
3 1 5 4 2

Thank you !!


4 1 5 3 2
1 2 3 4 5


1 2 3 4 5 6
-1 0 1 2 3 4 5 6


1 2 3 4 5 0 6 -1
-1 2 3 4 5 1 6 0
-1 0 3 4 5 2 6 1
-1 0 2 4 5 3 6 1
-1 0 2 1 5 4 6 3
-1 0 2 1 5 4 6 3