Grothendieck Polynomials of Fireworks and Inverse Fireworks Permutations

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Abstract

We study the combinatorial structure of the set of monomials appearing in Grothendieck polynomials for special families of permutations, which also resulted in a direct formula for the top degree components of these polynomials. This is an extended abstract of [1] and [2].

Motivation

Grothendieck polynomials $\mathfrak{G}_w$ were introduced by Lascoux and Schützenberger to represent K-classes of structure sheaves of Schubert varieties. They are inhomogeneous, and their lowest-degree component is the Schubert polynomial $\mathfrak{S}_w$. Their Newton polytopes, formed by their monomial supports, are conjectured to have nice properties. Monical, Tokcan, and Yong [3] initiated the study of the *saturated Newton polytope (SNP)* property of polynomials arising in algebraic combinatorics and conjectured that $\mathfrak{G}_w$ has SNP. Huh, Matherne, Mészáros, and St. Dizier [4] later conjectured that the homogenized Grothendieck polynomial $\widetilde{\mathfrak{G}}_w$ has *M-convex* support. Motivated by these conjectures, we study the monomial support of Grothendieck polynomials.

Conjectures on Support

For a permutation $w \in S_n$, Mészáros, Setiabrata, and St. Dizier [5] gave a series of conjectures on the support of its Grothendieck polynomial.

- Conjecture 1. If $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ and $|\alpha| < \deg \mathfrak{G}_w$, then there exists $\beta \in \mathrm{supp}(\mathfrak{G}_w)$ with $\alpha < \beta$.
- Conjecture 2. If $\alpha \in \mathrm{supp}(\mathfrak{G}_w)$ and $|\alpha| < \deg \mathfrak{G}_w$, then there exists $\beta \in \mathrm{supp}(\mathfrak{G}_w)$ with $\alpha < \beta$ and $|\alpha| + 1 = |\beta|$.
- Conjecture 3. If $\alpha, \gamma \in \mathrm{supp}(\mathfrak{G}_w)$, then $[\alpha, \gamma] \subseteq \mathrm{supp}(\mathfrak{G}_w)$.

Together, these conjectures amount to the following conjecture:

- Conjecture 4. Let $\mathfrak{G}_w^{\mathrm{top}}$ be the highest degree part of $\mathfrak{G}_w$, then

$$\mathrm{supp}(\mathfrak{G}_w) = \bigcup_{\substack{\alpha \in \mathrm{supp}(\mathfrak{S}_w) \\ \beta \in \mathrm{supp}(\mathfrak{G}_w^{\mathrm{top}})}} [\alpha, \beta].$$

M-convexity of $\widetilde{\mathfrak{G}}_w$ implies Conjecture 3 for $\mathfrak{G}_w$ but not Conjecture 4.
Previously, Conjecture 4 is only known for vexillary permutations [6], and M-convexity is only known for vexillary permutations [6] and permutations with multiplicity-free Schuberts [7].

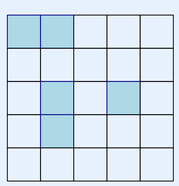
Fireworks and Inverse Fireworks

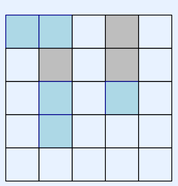
A permutation w is called *fireworks* if the initial terms of the decreasing runs of w are increasing. Equivalently, a permutation is fireworks if it is 3_12-avoiding. A permutation w is called *inverse fireworks* if w^{-1} is fireworks.

$$w = \underline{3}1\underline{6}7542 \quad \text{is fireworks,}$$
$$w^{-1} = 2716534 \quad \text{is inverse fireworks.}$$

These two families naturally appear in the study of top-degree Grothendieck polynomials.

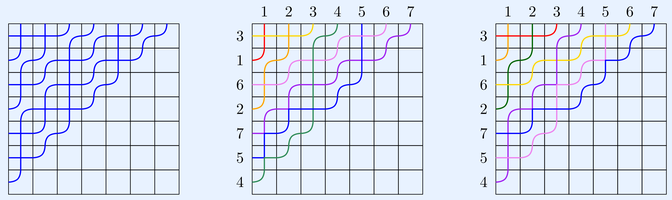
- Pechenik, Speyer, and Weigandt [8] showed that for all $w \in S_n$, there is a unique inverse fireworks permutation $\pi(w)$ with $\mathfrak{G}_w^{\mathrm{top}} = c_w \cdot \mathfrak{G}_{\pi(w)}^{\mathrm{top}}$ for some $c_w \in \mathbb{Z}$. In fact, $c_w = |\mathfrak{G}_{\pi(w^{-1})}^{\mathrm{top}}(1, \ldots, 1)|$. Therefore, to study $\mathfrak{G}_w^{\mathrm{top}}$ for any permutation, it suffices to study $\mathfrak{G}_w^{\mathrm{top}}$ for w inverse fireworks.
- When w is fireworks, $\mathfrak{G}_w^{\mathrm{top}}$ consists of a single monomial. Specifically, it is a positive integer multiple of $x^{\mathrm{wt}(\overline{D(w)})}$, where $\overline{D(w)}$ denotes the up-closure of the Rothe diagram $D(w)$ of w.





Pipe dreams

Grothendieck polynomials can be computed via the *pipe dream* formula [9, 10]. A pipe dream can be traced in two ways: from top to left or from left to top. These traces may look different, but they encode the same permutation.



Let $\mathrm{PD}(w)$ be the set of pipe dreams for w. Then the Grothendieck polynomials and homogenized Grothendieck polynomials can be defined as

$$\mathfrak{G}_w(\mathbf{x}) = \sum_{P \in \mathrm{PD}(w)} (-1)^{|\mathrm{wty}(P)| - \ell(w)} \prod_{(i,j) \in \mathrm{wty}(P)} x_i,$$

$$\widetilde{\mathfrak{G}}_w(\mathbf{x}, z) = \sum_{P \in \mathrm{PD}(w)} \left(z^{\deg(\mathfrak{G}_w) - |\mathrm{wty}(P)|} \cdot (-1)^{|\mathrm{wty}(P)| - \ell(w)} \prod_{(i,j) \in \mathrm{wty}(P)} x_i \right).$$

Theorem 1

For fireworks w,

$$\mathrm{supp}(\mathfrak{G}_w) = \bigcup_{\alpha \in \mathrm{supp}(\mathfrak{S}_w)} [\alpha, \mathrm{wt}(\overline{D(w)})] = \sum_{i=1}^{n} P_{sp}(\mathrm{SM}(D_i(w))).$$

where $P_{sp}(\mathrm{SM}(D_i(w)))$ is the *Schubert spanning set polytope* of the i^{th} column of $D(w)$.
In particular,

- $\widetilde{\mathfrak{G}}_w$ has M-convex support,
- $\mathfrak{G}_w$ has SNP,
- each degree component has M-convex support, and
- $|\mathrm{supp}(\mathfrak{G}_w^{(c)})|$ forms a log-concave sequence in c.

Theorem 2

Theorem: Let w be fireworks. Let $\pi(w)$ be the layered permutation whose block sizes are the lengths of the decreasing runs of w. Then

$$\mathrm{supp}(\mathfrak{G}_{\pi(w)}) \supseteq \mathrm{supp}(\mathfrak{G}_w).$$

Thus, layered permutation has maximal Grothendieck support among fireworks permutations.

References

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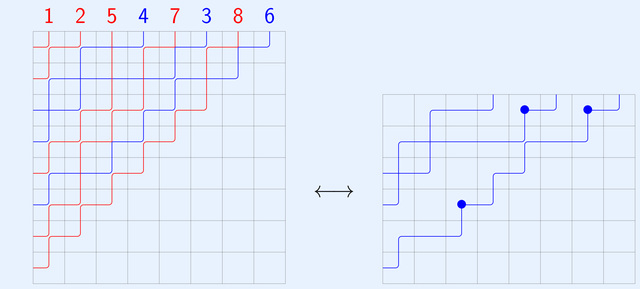
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Marked Vertical-less Pipe Dreams

Removing left to right maximum pipes from the left to top trace yields a bijection between pipe dreams and *marked vertical-less pipe dreams* $\mathrm{MVPD}(w)$ for any permutation. From this perspective, we are able to define local moves called *droop-moves* for pipe dreams.



For w inverse fireworks, $P \in \mathrm{MVPD}(w)$ is highest weight if and only if P has no  or , and the first column only consists of  and , yielding *bumpless vertical-less pipe dreams* BVPDs.

$$\begin{array}{ccc} \mathrm{PD}(w) & \longleftrightarrow & \mathrm{MVPD}(w) \\ \uparrow & & \uparrow \\ \widehat{\mathrm{PD}}(w) & \longleftrightarrow & \widehat{\mathrm{MVPD}}(w) \xleftarrow{Inv.Fire} \mathrm{BVPD}(w) \end{array}$$

Key: MVPDs provide local droop moves controlling support growth. BVPDs provide a direct combinatorial formula for the top Grothendieck polynomial.

Theorem 3

For a MBPD whose weight does not have maximal degree, we may apply a sequence of droop moves to obtain a new MBPD with a higher degree, thus proving Conjecture 2 for inverse fireworks permutations.

If $\mathbf{x}^\alpha$ appears in $\mathfrak{G}_w$ and $\deg(\mathbf{x}^\alpha) < \deg(\mathfrak{G}_w)$, then there exists i such that $x_i \mathbf{x}^\alpha$ also appears in $\mathfrak{G}_w$.

Theorem 4

A BVPD consists of the tiles , , , , and . It yields the first direct combinatorial formula for $\mathfrak{G}_w^{\mathrm{top}}$.

$$\mathfrak{G}_w^{\mathrm{top}} = |\mathrm{BVPD}(\pi(w^{-1}))| \sum_{P \in \mathrm{BVPD}(\pi(w))} \mathbf{x}^{\mathrm{wt}(P)}$$

where $\pi(w)$ is the inverse fireworks permutation associated with w in [8] and $\mathrm{wt}(P)$ is the number of , , and  in each row of P.

For $w = 165234$, we have the following 6 BVPDs.

